Foundations of Game Theory for Electrical and Computer Engineering

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Incomplete Information Games!

BAYESIAN NASH EQUILIBRIUM

Content

- Incomplete Information Games: Definitions
- Bayesian Nash Equilibrium
- Sheriff's Dilemma: An Example

Complete vs Incomplete Information Games

- In **complete information** games, everyone knows:
  - The number of players
  - The actions available to each player
  - The payoff associated with each action vector

- Note: These are still valid assumptions for imperfect information game, Why?

Complete vs Incomplete Information Games

- In **incomplete information (Bayesian)** games,
  - We represent players' uncertainties about the very game being played
  - This uncertainty is represented as a probability distribution over a set of possible games

- We make two assumptions:

1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
2. The beliefs of the different agents are posteriors, obtained by conditioning a common prior on individual private signals.

Bayesian Game Definition I: Information Sets

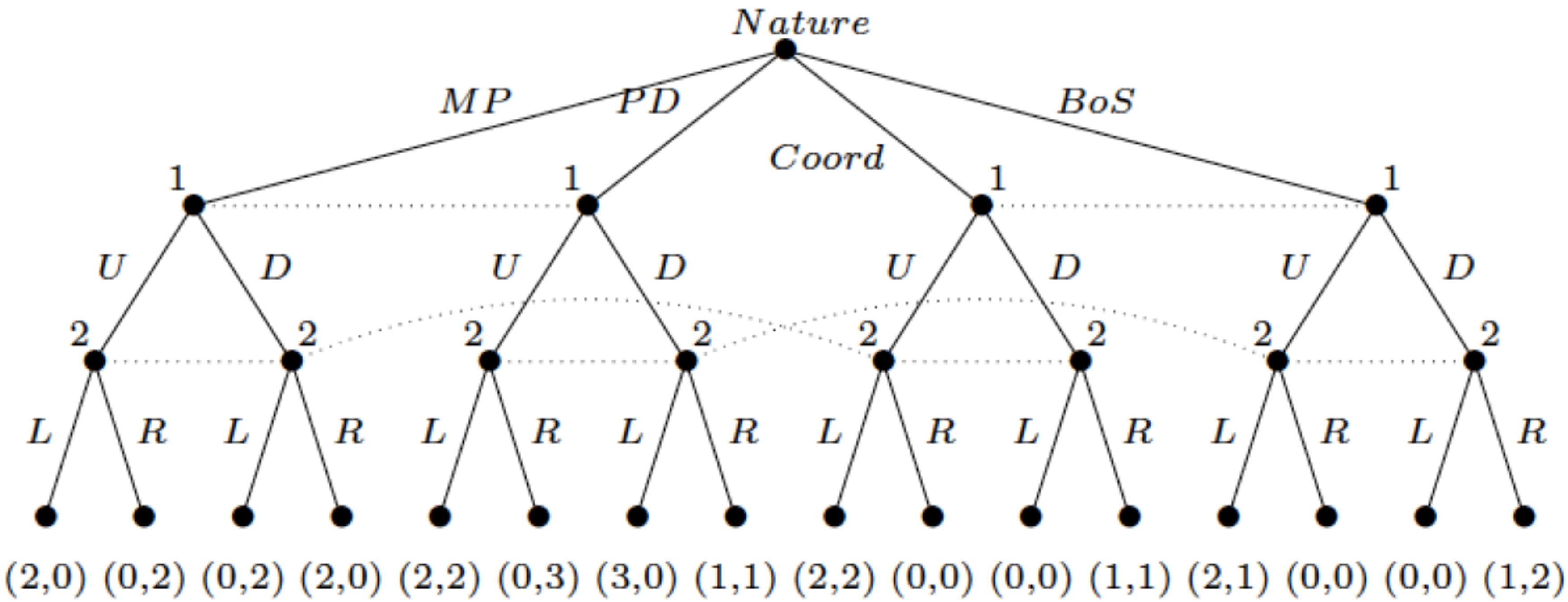
Definition 6.3.1 (Bayesian game: information sets) *A* Bayesian game *is a tuple* (N, G, P, I) *where:*

- N *is a set of agents;*
- G *is a set of games with* N *agents each such that if* $g, g' \in G$ *then for each agent* $i \in N$ *the strategy space in* g *is identical to the strategy space in* g'*;*
- $P \in \Pi(G)$ *is a common prior over games, where* $\Pi(G)$ *is the set of all probability distributions over* G*; and*
- $I = (I_1, ..., I_N)$ *is a tuple of partitions of* G*, one for each agent.*

Example

	$I_{2,1}$	$I_{2,2}$
$I_{1,1}$	MP $\begin{array}{\|c\|c\|}\hline 2,0 & 0,2 \\ \hline 0,2 & 2,0 \\ \hline \end{array}$ $p = 0.3$	PD $\begin{array}{\|c\|c\|}\hline 2,2 & 0,3 \\ \hline 3,0 & 1,1 \\ \hline \end{array}$ $p = 0.1$
$I_{1,2}$	Coord $\begin{array}{\|c\|c\|}\hline 2,2 & 0,0 \\ \hline 0,0 & 1,1 \\ \hline \end{array}$ $p = 0.2$	BoS $\begin{array}{\|c\|c\|}\hline 2,1 & 0,0 \\ \hline 0,0 & 1,2 \\ \hline \end{array}$ $p = 0.4$

Bayesian Game Definition II: Extensive Form



Bayesian Game Definition III: Epistemic Types

Definition 6.3.2 (Bayesian game: types) *A* Bayesian game *is a tuple* (N, A, Θ, p, u) *where:*

- N *is a set of agents;*
- $A = A_1 \times \cdots \times A_n$, *where* A_i *is the set of actions available to player* i*;*
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$, *where* Θ_i *is the type space of player* i*;*
- $p : \Theta \mapsto [0, 1]$ *is a common prior over types; and*
- $u = (u_1, \ldots, u_n)$, *where* $u_i : A \times \Theta \mapsto \mathbb{R}$ *is the utility function for player* i.

A Few Words About Types

- The type of agent encapsulates all the information possessed by the agent that is not common knowledge, e.g.,
- The agent's knowledge of his private payoff function
- His beliefs about other agents' payoffs
- Their beliefs about his own payoff
- Any other higher-order beliefs …

Example for Definition III

a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

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Bayesian Nash Equilibrium

- We look for a plan of action for each player as a function of types that maximize **each type's expected** utility
  - **Expecting** over the **actions of other players**
  - **Expecting** over the **types of other players**

Strategies in Bayesian Game

- **Pure Strategy** is a choice of a pure action for player i as a function of his/her type

$$s_i : \Theta_i \rightarrow A_i$$

- **Mixed Strategy** is a mixed action for player i as a function of his/her type

$$s_i : \Theta_i \rightarrow \Pi(A_i)$$

- In other words: $s_i(a_i \mid \theta_i)$
  - The probability under mixed strategy s_i that agent i plays action a_i, given that i's type is θ_i

Expected Utility

- **Ex-ante**
  - The agent knows nothing about anyone's actual type
- **Ex-Interim**
  - An agent knows her own type but not the types of the other agents
- **Ex-poste**
  - The agent knows all agents' type

Ex-Post Expected Utility

The agent knows all agents' type, then for a given strategy profile s

$$EU_i(s,\theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a,\theta)$$

Ex-Ante Expected Utility

The agent knows nothing about anyone's actual type, then for a given strategy profile s

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i)$$

Ex-Interim Expected Utility

An agent knows her own type but not the types of the other agents, then for a given strategy profile s

$$EU_i(s,\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a\in A} \left(\prod_{j\in N} s_j(a_j|\theta_j)\right) u_i(a,\theta_{-i},\theta_i)$$

Bayesian Equilibrium

- A Bayesian Equilibrium is a mixed strategy profile s that satisfies

$$\forall i \quad s_i \in BR_i(s_{-i})$$

- Where:

$$BR_i(s_{-i}) = \arg\max_{s_i' \in S_i} EU_i(s_i', s_{-i})$$

BNE

- Players choose strategies to maximize their payoffs in response to others accounting for:
  - Strategic uncertainty about how others will play
  - Payoff uncertainty about the value to their actions

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Sheriff's Dilemma

- **Players:** a Sheriff and an armed suspect
- Sheriff must decide to shoot or not
- The suspect is either criminal with probability p or not with probability 1-p
- The sheriff would rather shoot if the suspect shoots, but not if the suspect does not
- The criminal would rather shoot even sheriff does not
- The innocent suspect would rather not shoot even if the sheriff shoots

Sheriff's Dilemma

Good: 1-p

Suspect \ Sheriff	Shoot	Not
Shoot	-3, -1	-1, -2
Not	-2, -1	0, 0

Bad: p

Suspect \ Sheriff	Shoot	Not
Shoot	0, 0	2, -2
Not	-2, -1	-1, 1

Sheriff's Dilemma

- There are **two strictly dominated** strategies in two cases
- We need to calculate the expected payoffs for each action

$$Eu_s(Shoot) = -1(1-p)+0(p)$$
$$Eu_s(Not\text{-}Shoot) = 0(1-p)-2(p)$$

If $p>1/3$ ➔ Sheriff shoots
If $p<1/3$ ➔ Sheriff does not shoot
If $p=1/3$ ➔ Any mixture

John Nash (June 1928-May2015) [JSAC 2008]

FINAL WORDS

Nash Foreword
JSAC September 2008
Game Theory Applied to Computer and Communication Systems

I **am not well informed** personally about all of the various studies that are currently applying Game Theory to areas of engineering. But **I have seen**, for example, **interesting applications of Game Theory or also of Experimental Games studies to problems of traffic flow.**

Nash Foreword
JSAC September 2008
Game Theory Applied to Computer and Communication Systems

For example, experimental game theory can employ human subjects to test out traffic behavior when the human drivers are responsible for the travel route choices that are made.

Other studied areas include cases **of resource allocation in wireless networks without any central control, packet routing in wireline networks, and the traffic flow of commands in computer circuits**.

Nash Foreword
JSAC September 2008
Game Theory Applied to Computer and Communication Systems

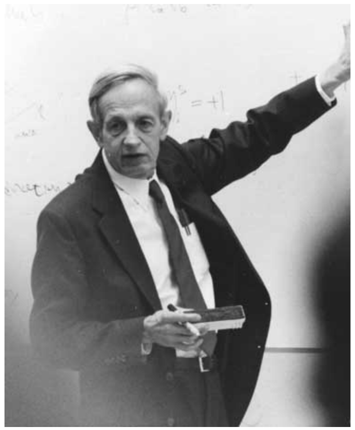
What will be called "Game Theory" in the future cannot be infallibly prophesied, I feel. For example, in current scientific nomenclature there are "astrophysics" and "cosmology" which are not quite the same but which are overlapping.

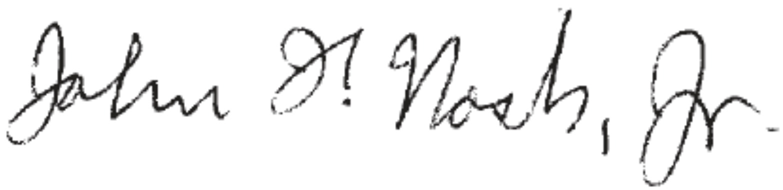
Nash Foreword
JSAC September 2008
Game Theory Applied to Computer and Communication Systems

As regards the future,
I cannot easily predict what the future extent of study and uses of "game theory" will be in part simply because the terminological conventions and usages can change.

Nash Foreword
JSAC September 2008
Game Theory Applied to Computer and Communication Systems

For example, it could become customary to describe studies as **"behavioral economics"** (instead of as "game theoretical"). Although it is not certain what "Game Theory" be called in the future, one can expect, in this twenty-first century, that the areas of application will continue with expansions and widenings.





John F. Nash, Jr.