بسم الله الرحمن الرحيم

Foundations of Game Theory for Electrical and Computer Engineering

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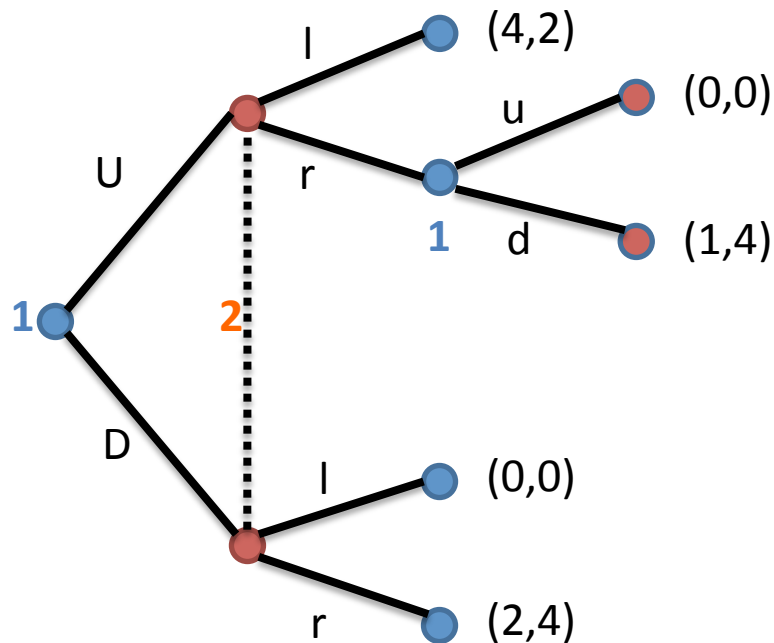
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Dynamic Games III

1. Solve Imperfect Information Games
2. Sub-game Definitions
3. Sub-game Perfect Nash Equilibrium (SPNE)
4. Don't Screw-Up Game
5. Match Maker Game
6. Applied Examples:
   a. Market Game
   b. Microsoft vs Mozilla
   c. TDMA Transmission

Solve Imperfect Information Games

- Before we analyze the game, let's figure out some basic facts
- How many information sets we have?
  - Player 2 has 1 information set
  - Player 1 has 2 information sets

- What are the strategies?
  - Player 1: **Uu, Ud,** *Du, Dd*
  - Player 2: **l, r**

Static Game Solution

Player 2

	l	r
Uu	4,2	0,0
Ud	4,2	1,4
Du	0,0	2,4
Dd	0,0	2,4
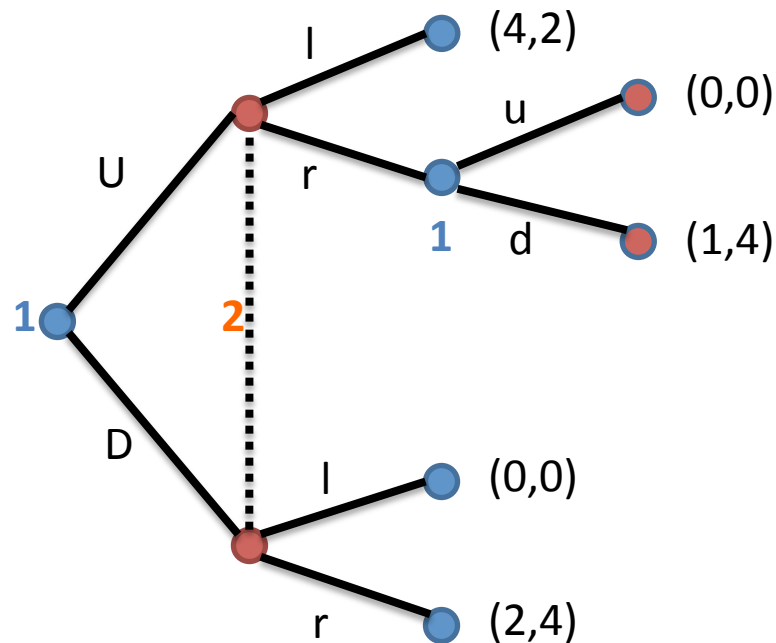
Player 1

- *Do you notice the redundancy here?*

- Let's find the NE of this game

Solutions: Nash Equilibria

Player 2

	l	r
Uu	4,2	0,0
Ud	4,2	1,4
Du	0,0	2,4
Dd	0,0	2,4

Player 1

(Uu,l)

(Du,r)

(Dd,r)

Dynamic Game Solution

sub-game
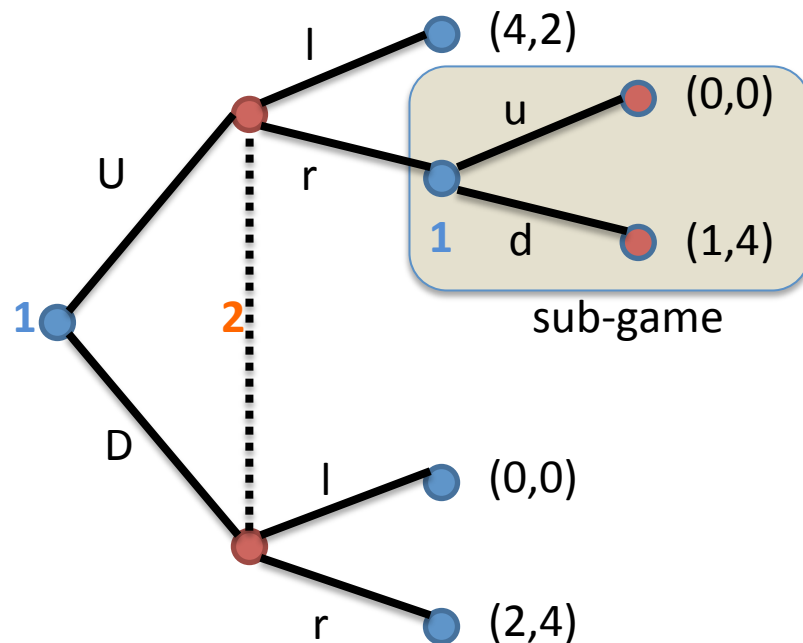
- Let's try to use **BI**
- Starting from the end, player 1 will choose *down*
- Then, although player 2 doesn't know where she is on the tree, she will notice that she's always better-off choosing *right*
- This implies that player 1 will then choose *down*

NE of Static Games vs BI

- Nash Equilibria:
    - (Uu,l) ➜ not compatible with BI
    - (Du,r) ➜ not compatible with BI
    - (Dd,r) ➜ **This is compatible with BI**

- We're not saying these are not NE, it's just that they are **inconsistent** with what we could predict with BI

➜ We need a new notion of solution, that is able to treat games that have both sequential moves and simultaneous moves

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3-Player Game
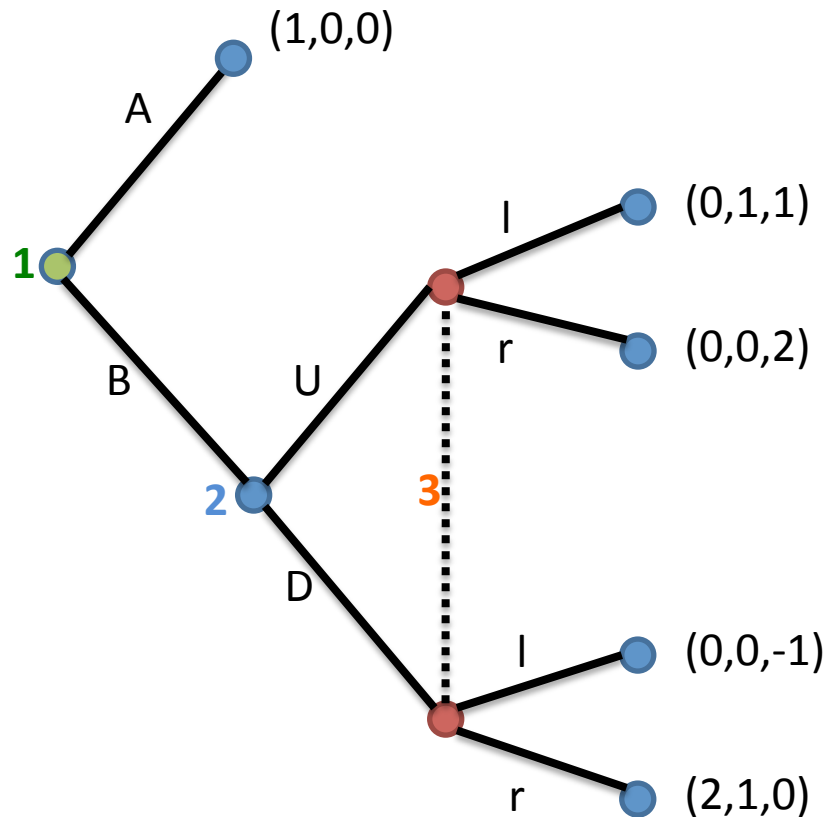
- We will model, in the next slide, the game as follows:
  – Player 1 chooses a **matrix**
  – Player 2 and 3 will play the game player 1 chose

Static Game Model

Player 1: A

3

	l	r
U	1,0,0	1,0,0
D	1,0,0	1,0,0

2

Player 1: B

3

	l	r
U	0,1,1	0,0,2
D	0,0,-1	2,1,0

2

- There are a lots of NE in this game!
  - E.g.: [A,U,l]
- *Question*: How can you check that it is a NE?
- *Question*: Does this NE make sense?

3-Player Dynamic Game

(1,0,0)

A

1

B

U

(0,1,1)

l

r

(0,0,2)

2

3

D

l

(0,0,-1)

r

(2,1,0)
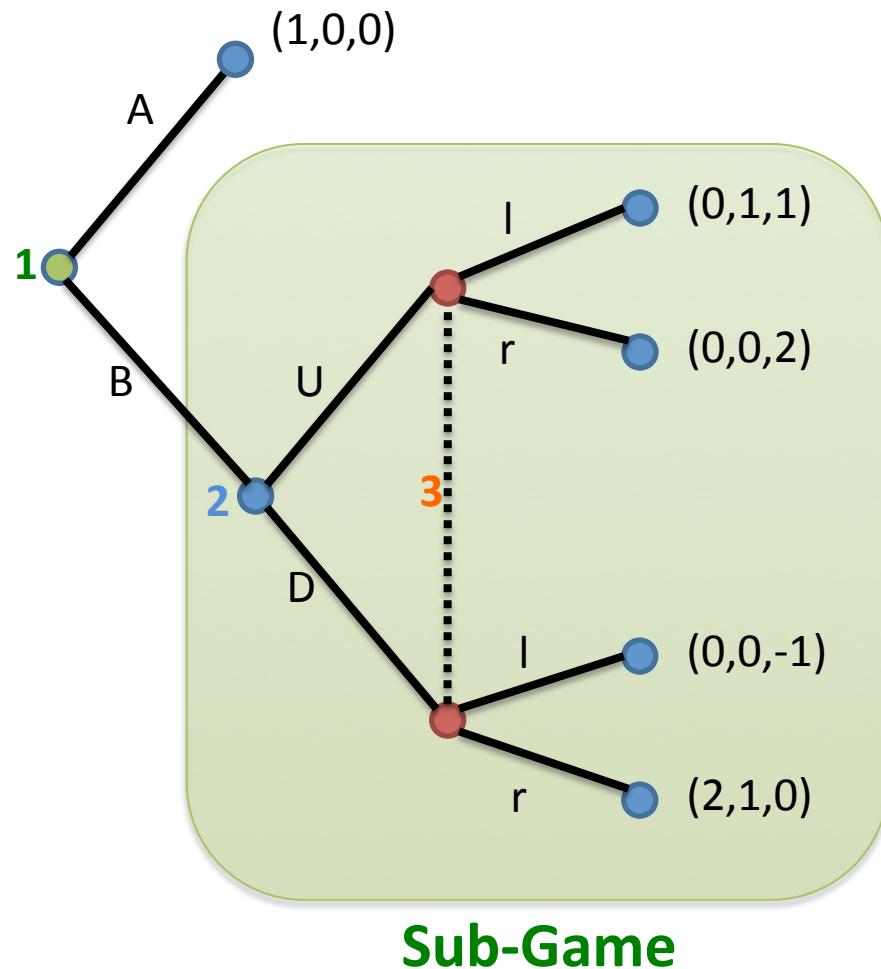
Sub-Game

- Let's have a look at the sub-game we identify in the game-tree
  - *Observation*: it involves only two players

Player 3

Player 2	l	r
U	1,1	0,2
D	0,-1	1,0

NE of our Sub-Game

Player 3

	l	r
U	1,1	0,2
D	0,-1	1,0

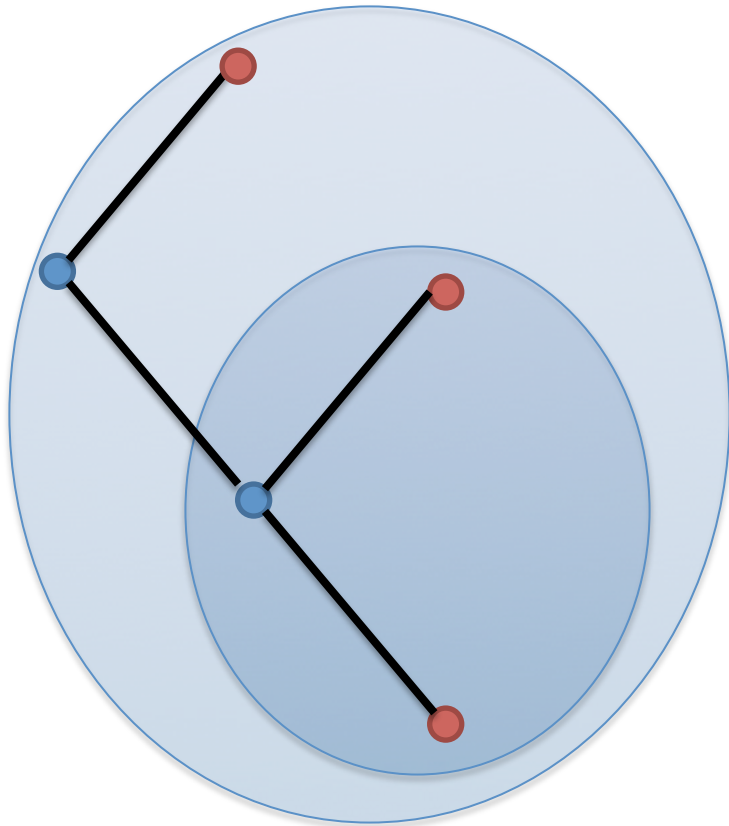**Player 2**

- What are the NE of this sub-game?
  - Notice that player 3 has a **dominant strategy**
  ➜NE = (D,r)
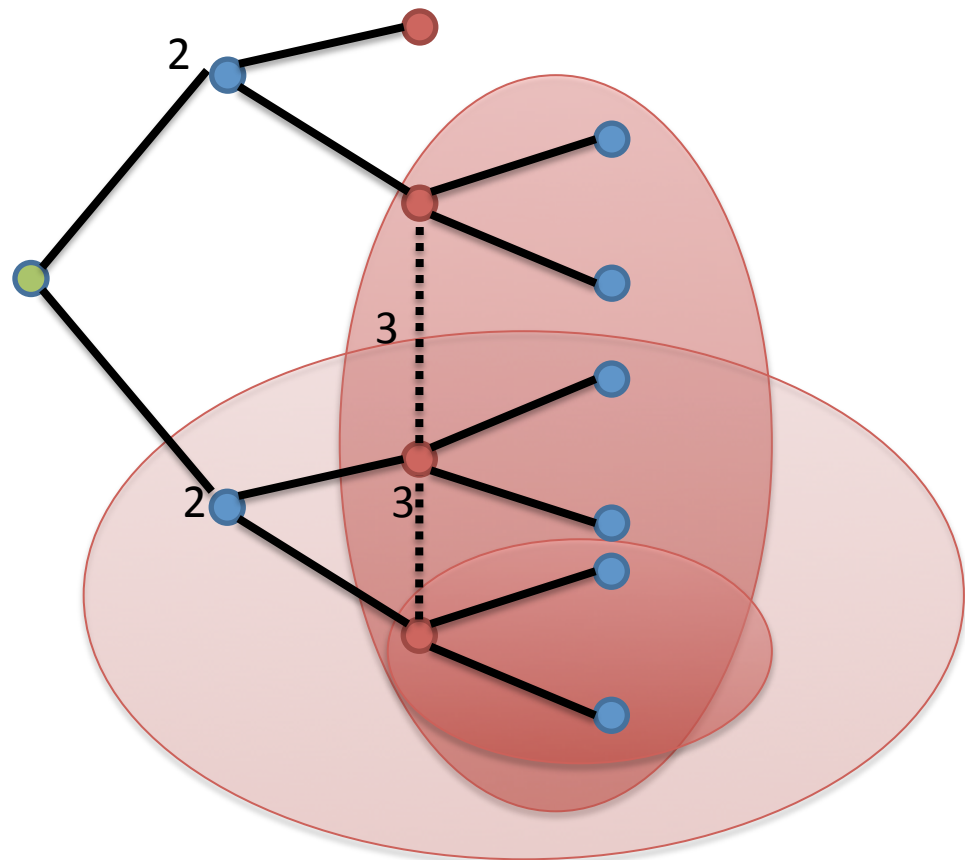- This new equilibrium clashes with the equilibrium we just found before!

Definition: Sub-Games

➢ A **sub-game** is a part of the game that looks like a game within the tree. It satisfies the three following properties:

1. It starts from a **single node**
2. It **comprises all successors** to that node
3. It **does not** break up any information set

Examples of Sub-Games

2 Sub-games

1 Sub-game

What do we want?

Let's find a way to rule out those Nash equilibria that instruct players down the tree to play in sub-games according to strategies that are not Nash equilibria.
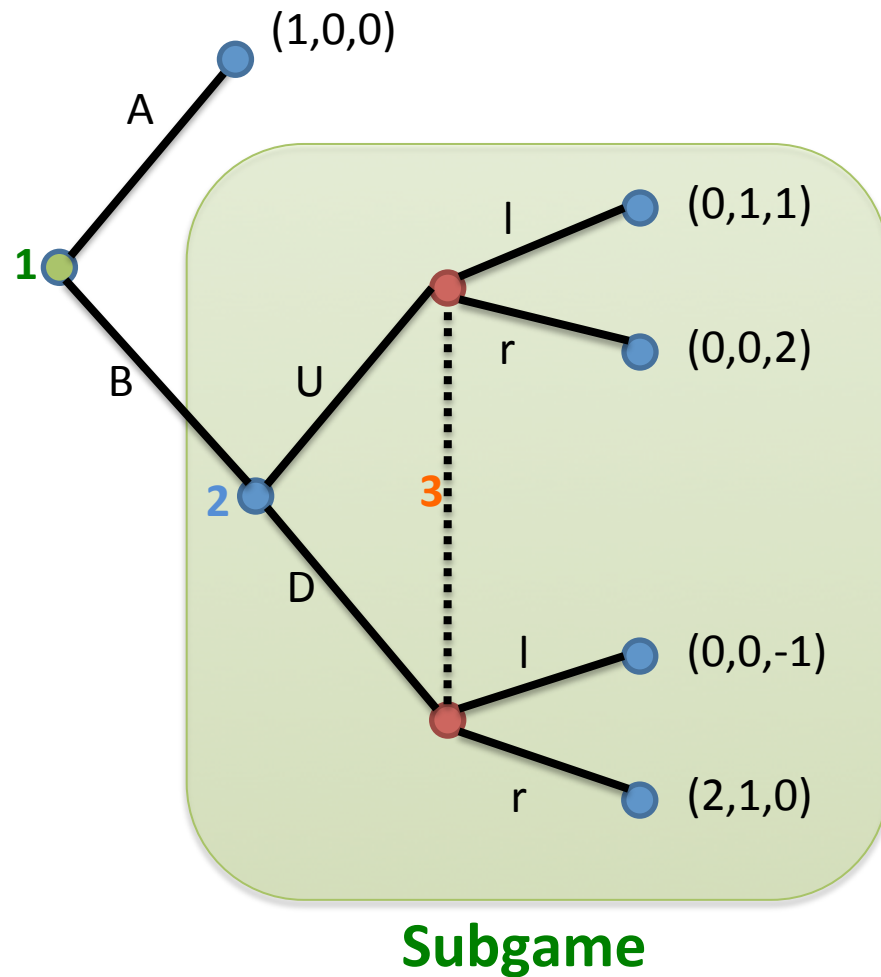
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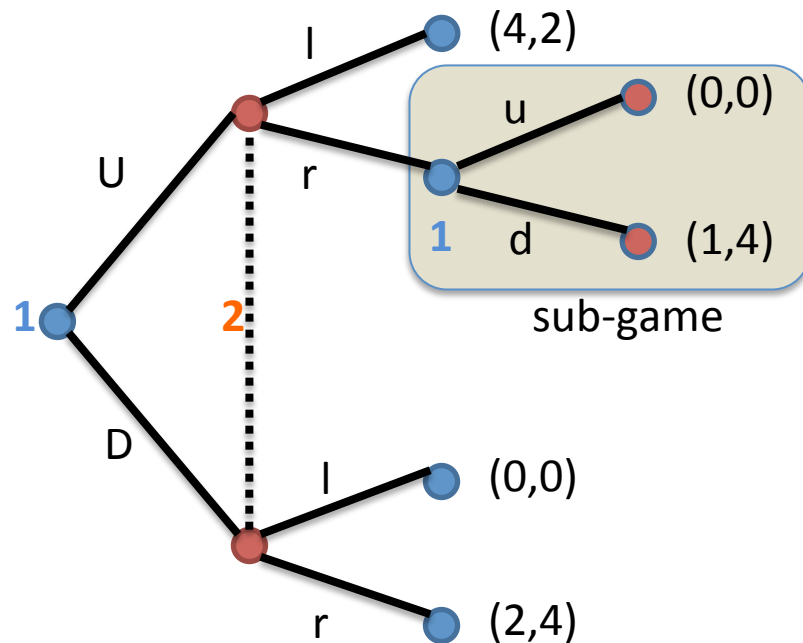# *Definition:*
Sub-game Perfect Equilibrium

A Nash Equilibrium $(s_1{}^*, s_2{}^*, \ldots, s_N{}^*)$ is a **Sub-Game Perfect Nash Equilibrium (SPNE)** if it induces a Nash Equilibrium in every sub-game of the game

SPNE: Example I

The **SPNE is (B,D,r),** because (D,r) is the only NE of the only subgame
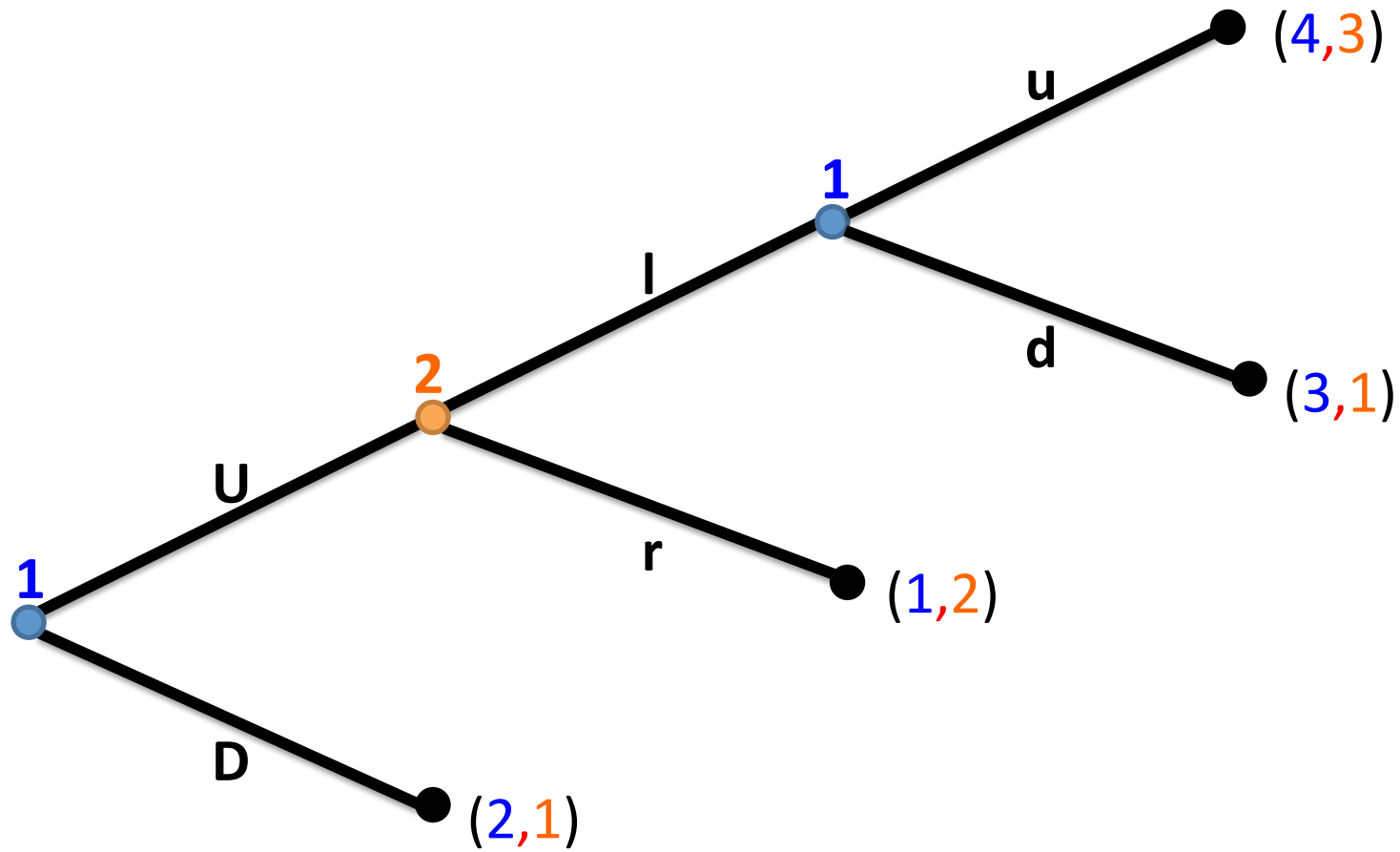
SPNE: Example II

The **SPNE is (Dd,r)** Because (d) is the only NE of the only sub-game
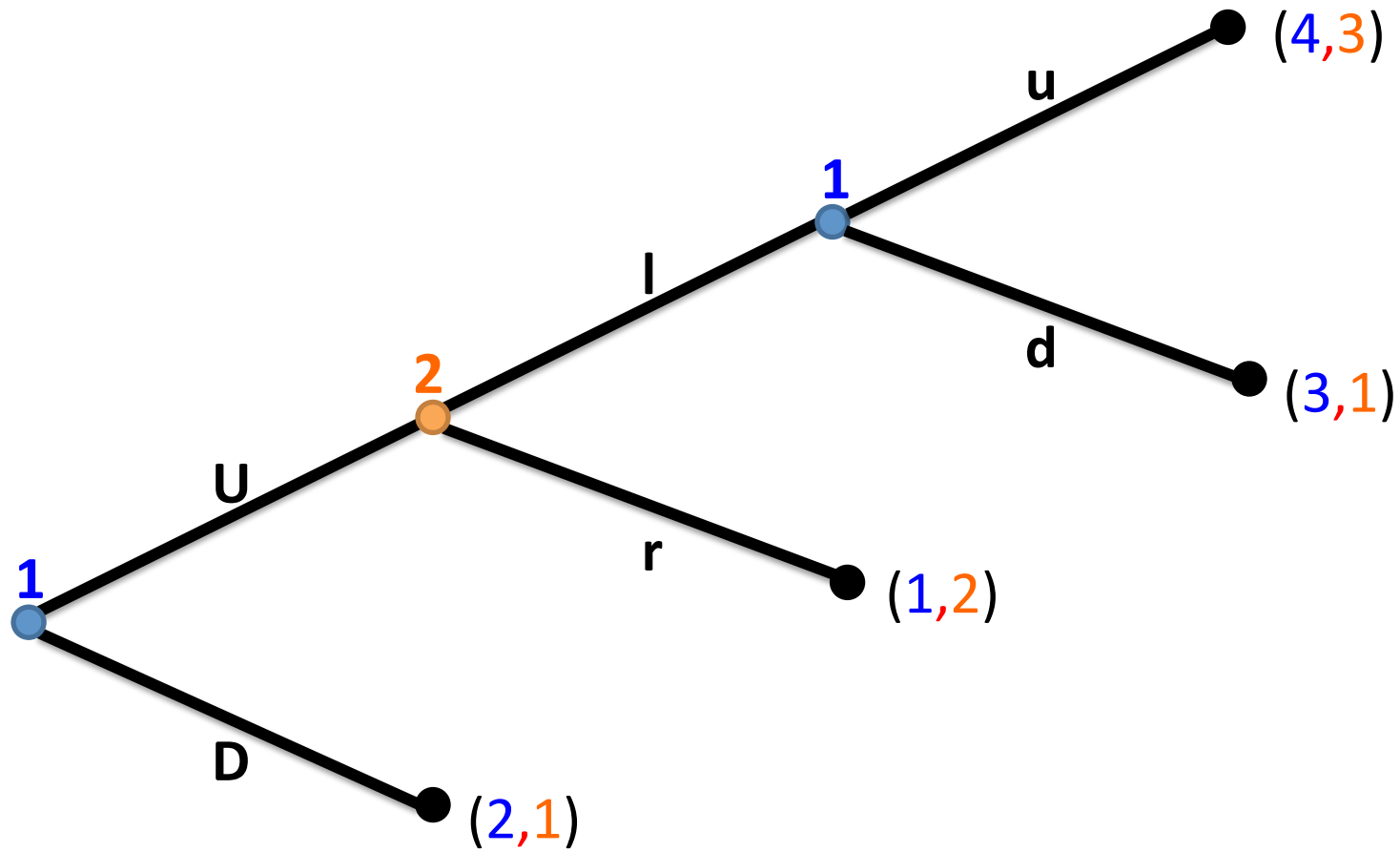
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Don't Screw Up

Let's Play This!

Solution: Backward Induction

Solution: Nash Equilibria

	l	r
Uu	4,3	1,2
Ud	3,1	1,2
Du	2,1	2,1
Dd	2,1	2,1

Player 1

Nash Equilibria:
1. **(Uu,l) ➔ compatible with BI**
2. (Du,r) ➔ not compatible with BI
3. (Dd,r) ➔ not compatible with BI

Subgame I
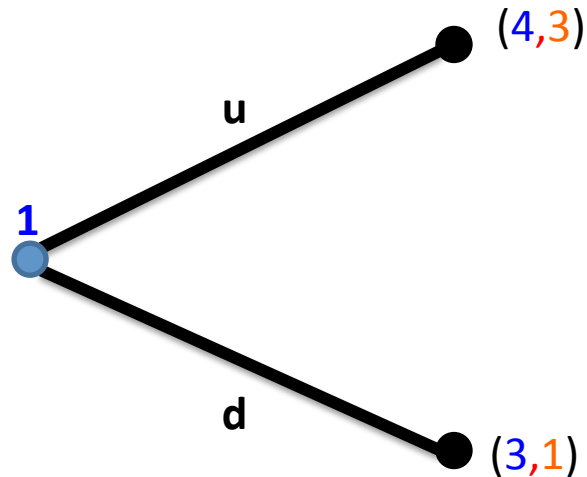
Player 1

u	(4),3
d	3,1

NE = (u)

(4,3)

u

1

d

(3,1)

(Uu,l)	(Du,r)	(Dd,r)
✓	✓	✗

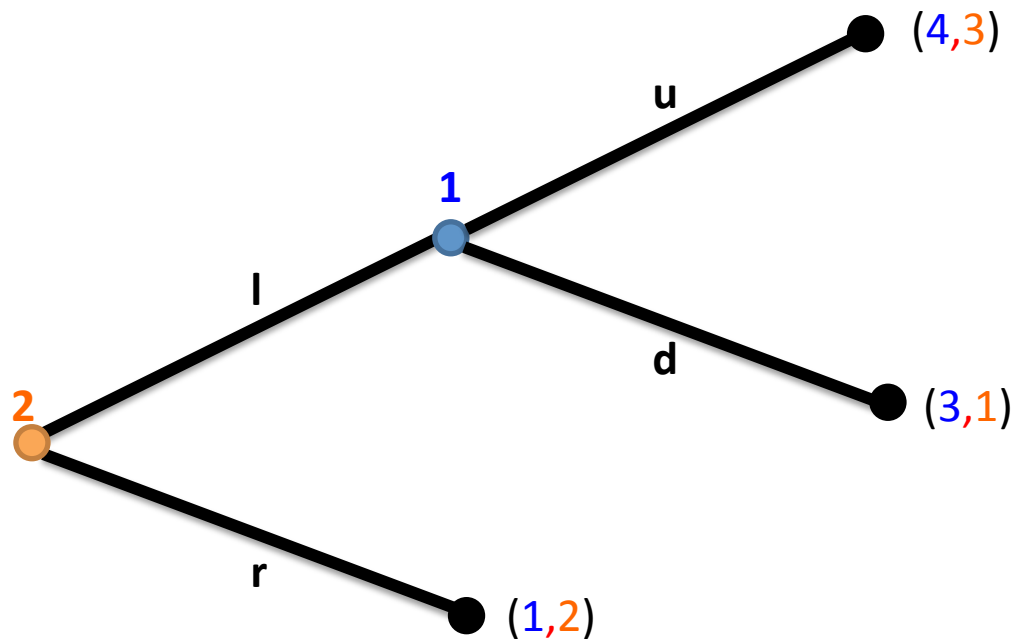So here (Dd,r) is eliminated since it induces non-Nash equilibrium play in this sub-game

Subgame 2

	l	r
u	4,3	1,2
d	3,1	1,2

Player 2 across top, Player 1 along side.

NE = (u,l) and (d,r)

(Uu,l)	(Du,r)	(Dd,r)
✓	✗	✓

So here (Du,r) is eliminated since it induces
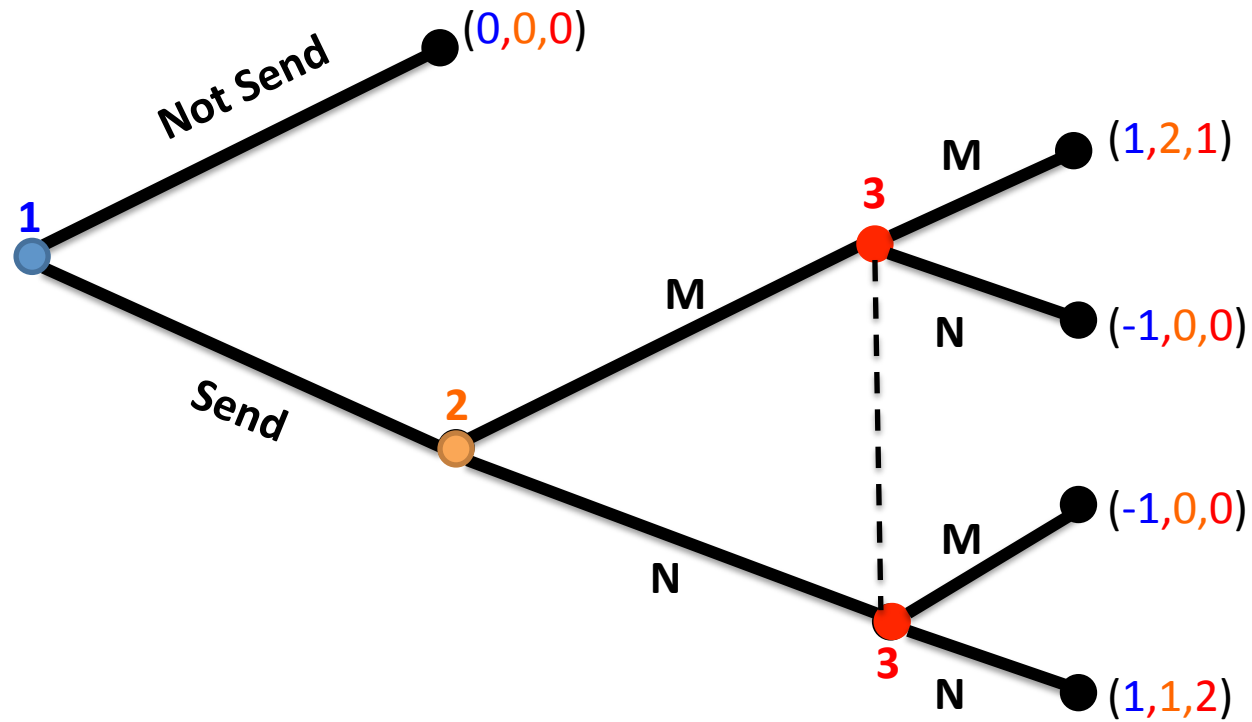non-Nash equilibrium play in this sub-game

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Solution: SPNE

- The only SPNE is (Uu,l)
  - This is the backward induction prediction

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Matchmaker Game

Sub-game Solution
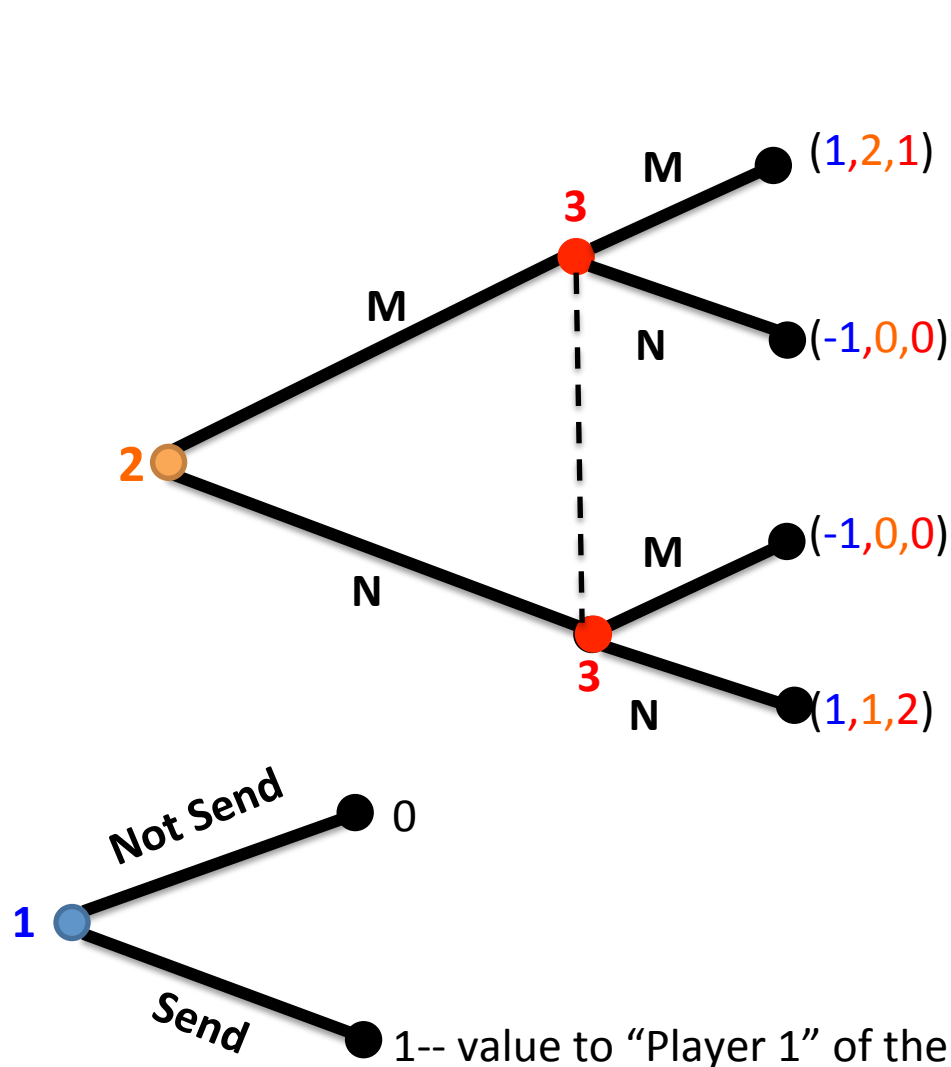
2	3 M	N
M	1,2,1	-1,0,0
N	-1,0,0	1,1,2

NE = (M,M) and (N,N)

1-- value to "Player 1" of the Nash equilibria in this sub-game

Solution SPNE

➢ There are two SPNE
  1. (Send,M,M)
  2. (Send,N,N)

Mixed Strategy Solution

- There is a Mixed NE
  - [(2/3,1/3),(1/3,2/3)]
- If Player 1 sends 2 and 3, then they meet with probability 2/9+2/9=4/9 and fail to meet with probability 5/9
- Player 1 expected payoff at this equilibrium is: 4/9(1) + 5/9 (-1) = -1/9

Not Send ● 0

1 ●
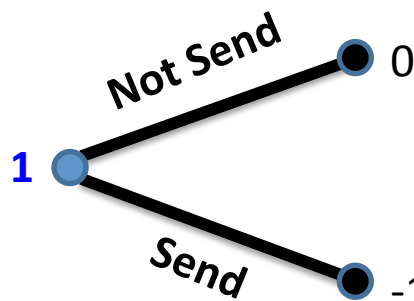
➔ SPNE= (Not Send, Mix NE, Mix NE)

Send ● -1/9-- value to "Player 1" of the Nash equilibria in the sub-game

Conclusions

- **Sub-game perfect equilibrium** implies **backward induction**

- Look for the Nash equilibria in each of the sub-games, roll the payoffs back up, and then see what the optimal moves are higher up the tree
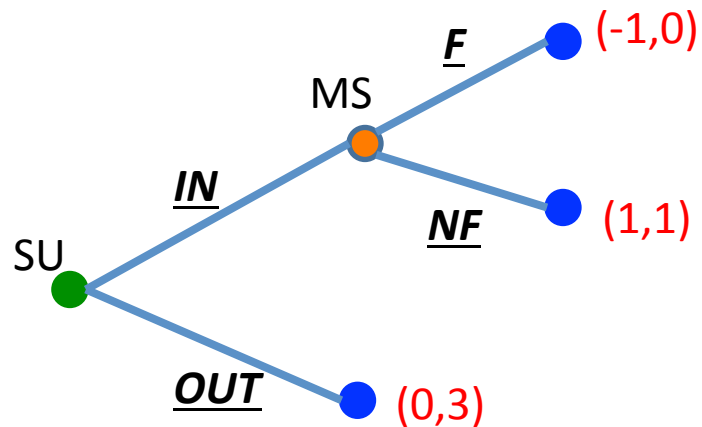
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A Market Game

- Assume there are two players

  – An incumbent monopolist (MicroSoft, MS) of O.S.

  – A young start-up company (SU) with a new O.S.

- The strategies available to SU are:
  Enter the market (**_IN_**) or stay out (**_OUT_**)

- The strategies available to MS are:
  Lower prices and do marketing (**_FIGHT_**) or stay put (**_NOT FIGHT_**)

A Market Game

- What should you do?

- Analyze the game with BI
- Analyze the normal form equivalent and find NE

A Market Game

SU \ MS	F	NF
IN	-1,0	1,1
OUT	0,3	0,3

Backward Induction

(IN, NF)

Nash Equilibrium

(IN, NF)
(OUT, F)

This is a NE, but relies on an **_incredible_** threat

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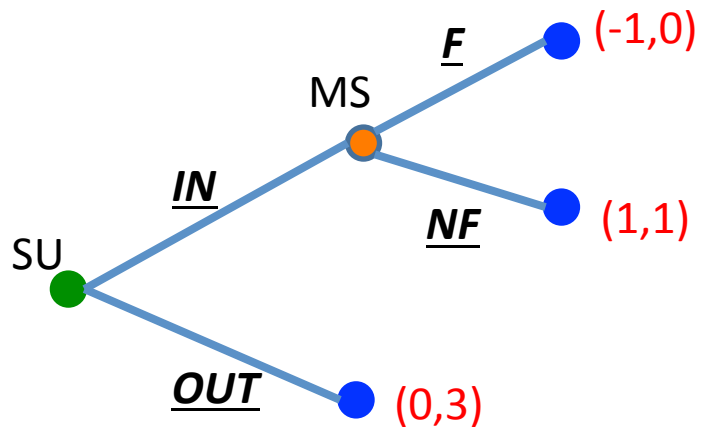   b. Microsoft vs Mozilla
   c. TDMA Transmission

Microsoft vs Mozilla: Game Tree

- Microsoft and Mozilla are deciding on adopting new browser technology (.net or java)
  - Microsoft moves first, then Mozilla makes its move

Microsoft

.net   java

Mozilla   Mozilla

.net   java   .net   java

3,1   1,0   0,0   2,2
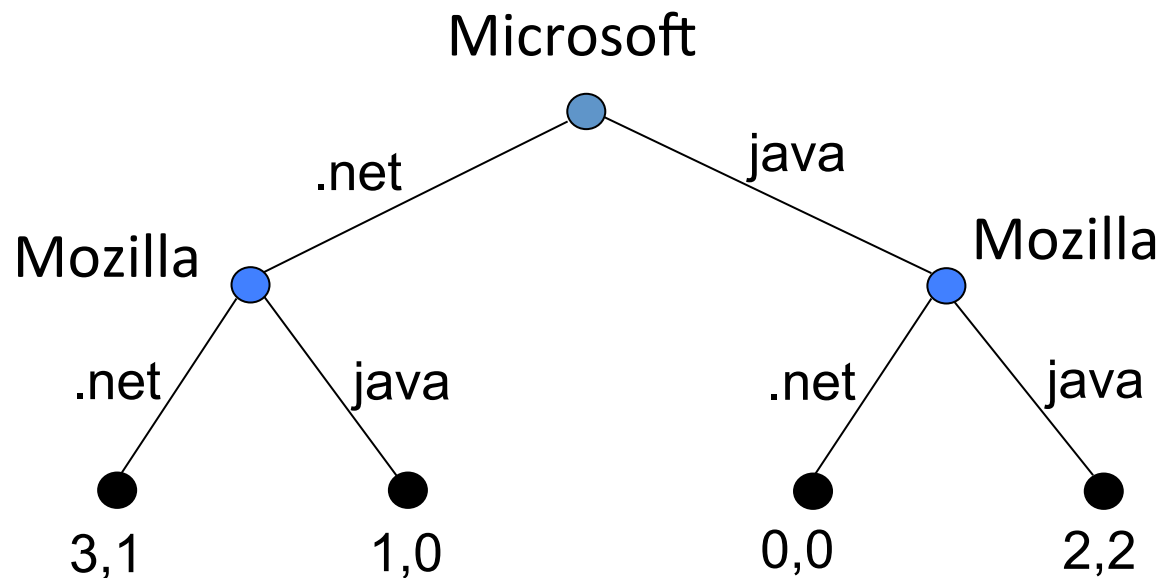
Converting to Matrix Game

Microsoft

.net   java

Mozilla   Mozilla

.net   java   .net   java

3,1   1,0   0,0   2,2
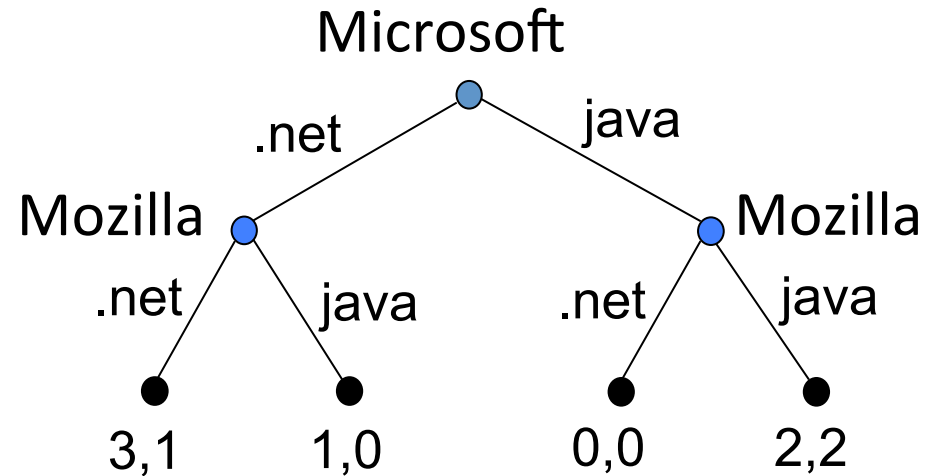
Mozilla

	.net, .net	.net, java	java,.net	java, java
Microsoft **.net**	3, 1	3, 1	1, 0	1, 0
java	0, 0	2, 2	0, 0	2, 2

NEP and Incredible Threats

Microsoft

.net   java

Mozilla

.net   java   .net   java

3,1   1,0   0,0   2,2

Mozilla

		.net, .net	.net, java	java, .net	java, java
Microsoft	**.net**	3, 1	3, 1	1, 0	1, 0
	java	0, 0	2, 2	0, 0	2, 2

NEP

incredible threat
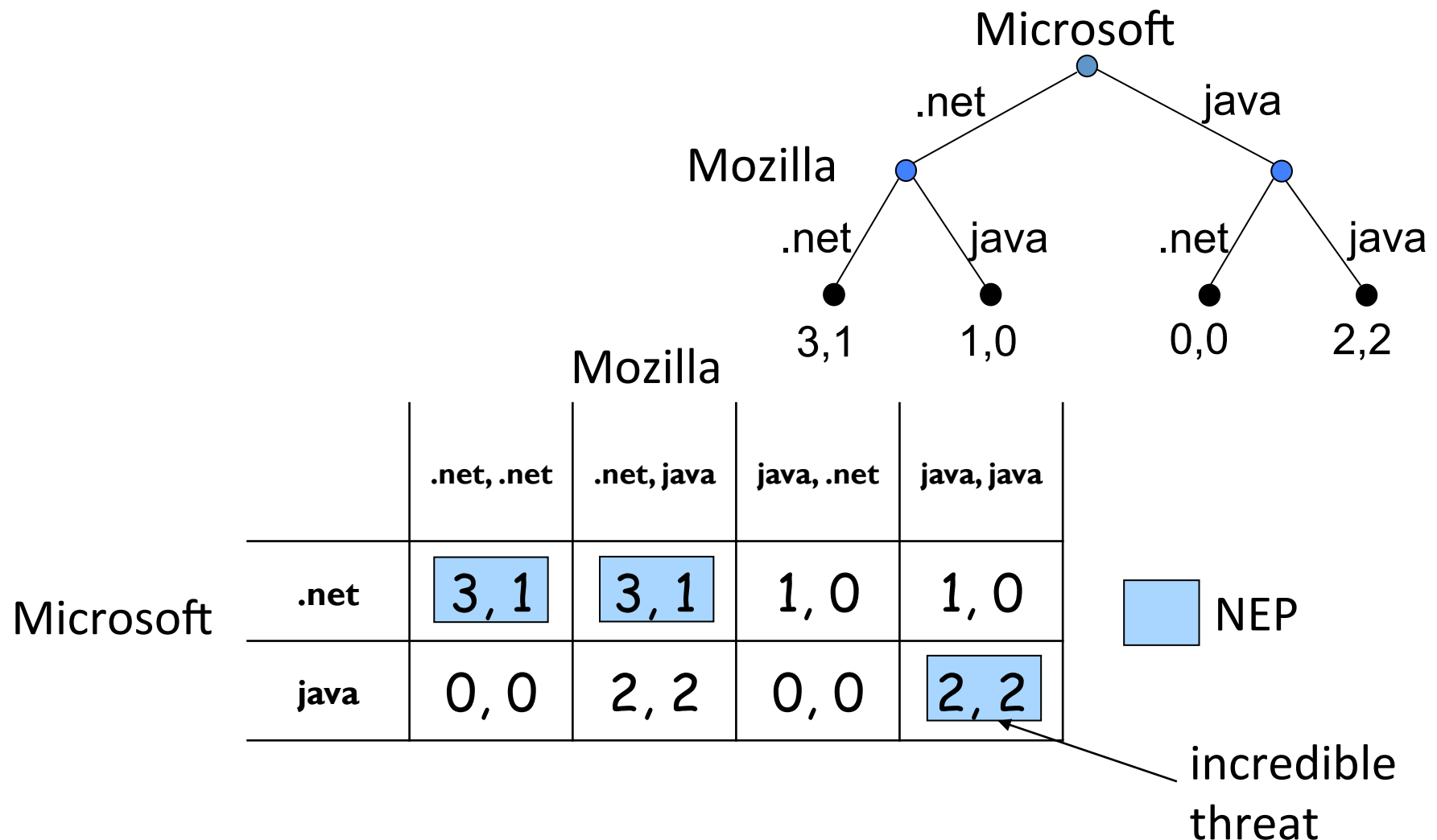
❏ Play "java no matter what" is not credible for Mozilla

  ○ if Microsoft plays .net then .net is better for Mozilla than java

Solving the Game (Backward Induction)

Microsoft

.net   java

Mozilla

.net   java   .net   java

3, 1   1, 0   0, 0   2, 2

Best strategy for
Mozilla: .net, java
(follow Microsoft)

Equilibrium
outcome

.net   java

3, 1   2, 2

Best strategy for
Microsoft: .net
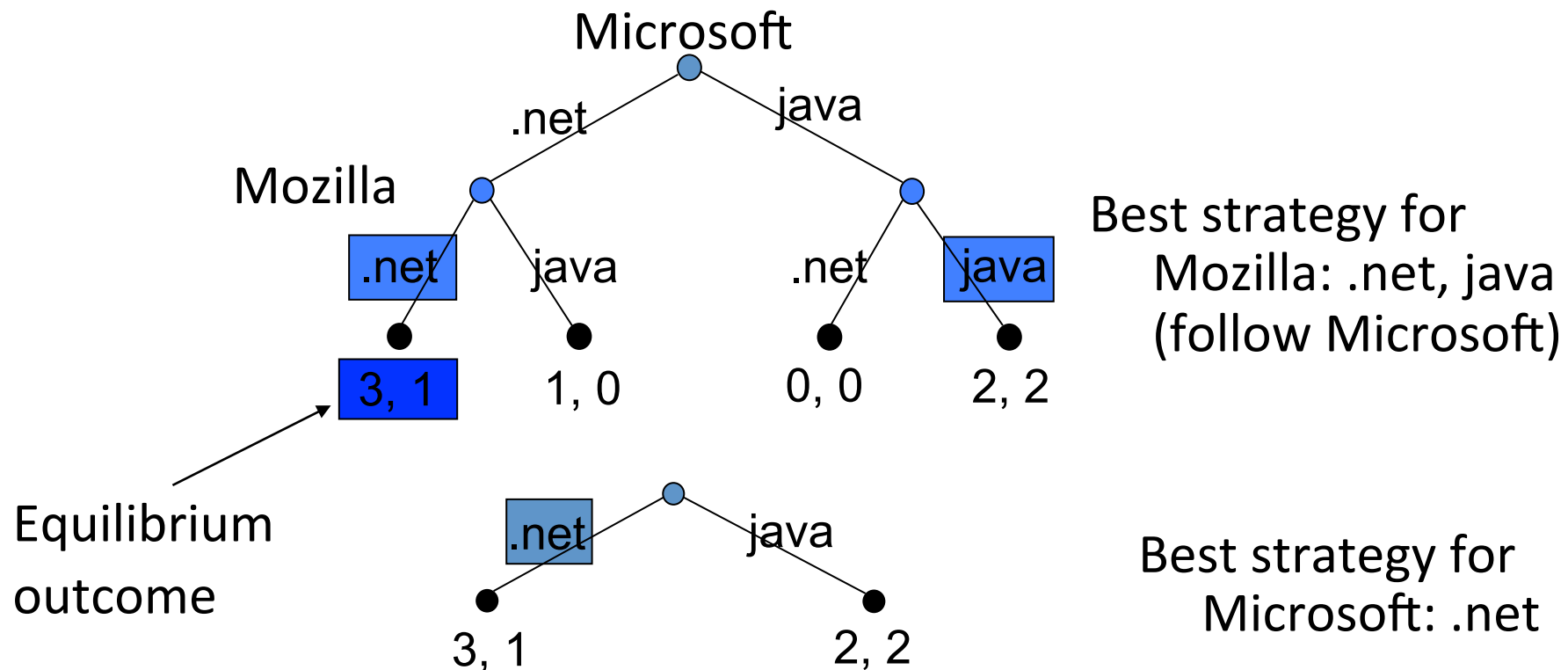
❒ Single NEP
Microsoft -> .net,   Mozilla -> .net, java

Kuhn's Theorem

Backward induction always leads to Nash Equilibrium in Sequential Games with Perfect Information

- Effective mechanism to remove "bad" NEP
  - incredible threats

Leaders and Followers

- What happens if Mozilla moves first?

Mozilla

.net ─── java

Microsoft   Microsoft

.net java   .net   java

1, 3   0,0   0,1   2,2

Mozilla: java
Microsoft: .net, java
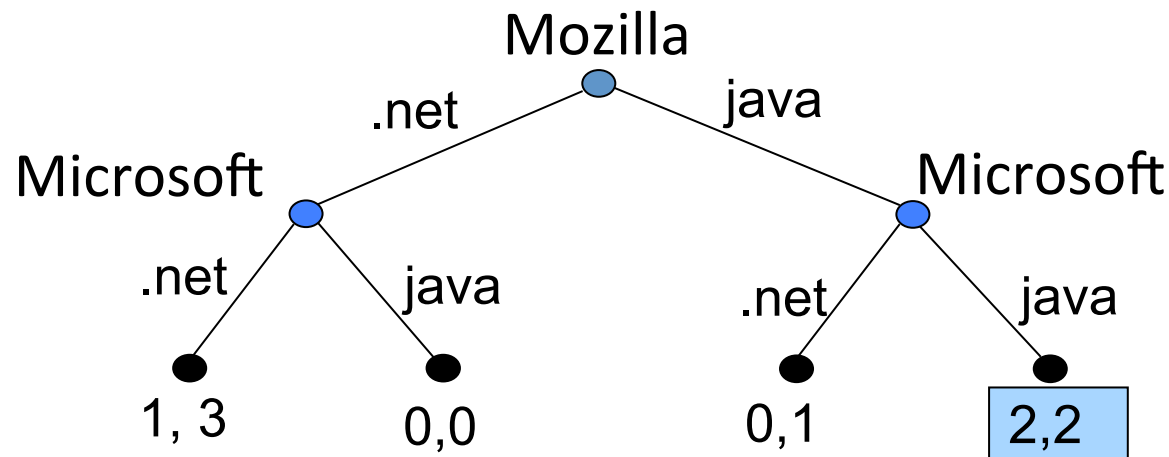
❒ NEP after backward induction:

❒ Outcome is better for Mozilla, worst for Microsoft
  ○ **incredible threat becomes credible!**
❒ 1st mover advantage
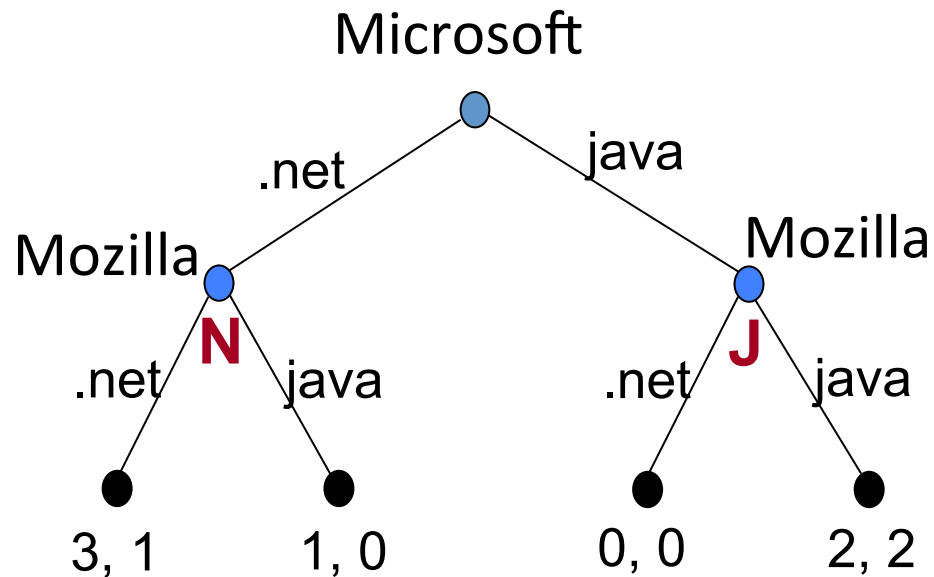 ○ Remember that it can also be a disadvantage…

The Sub-Games

- Definition: A sub-game is any sub-tree of the original game that also defines a proper game
  - includes all descendants of non-leaf root node
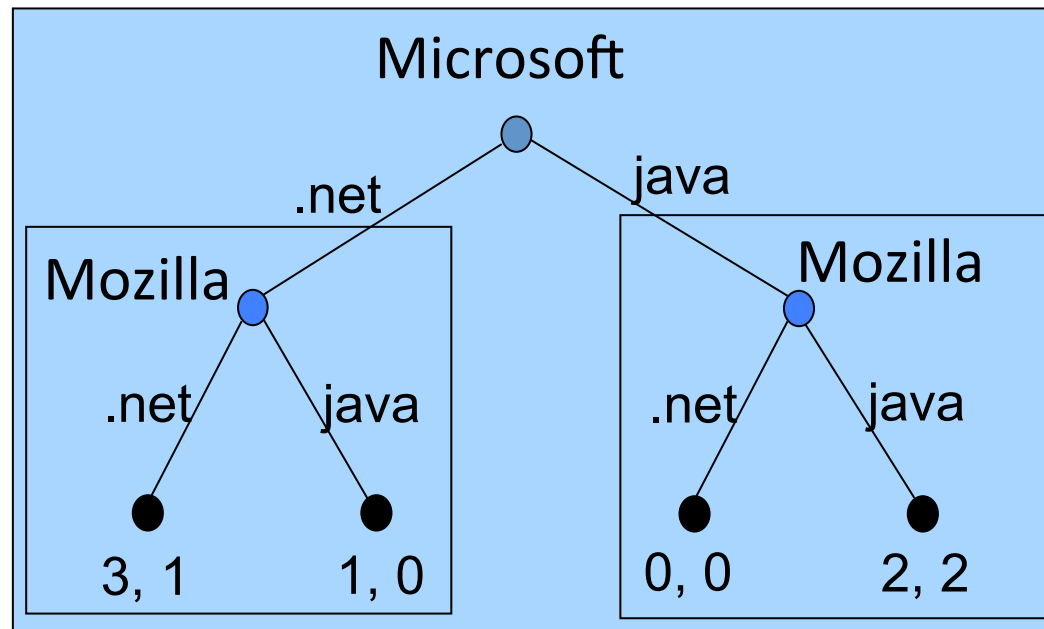
❑ 3 subtrees
  ○ full tree, left tree, right tree

Sub-game Perfect Nash Equilibrium

Microsoft

.net — java

Mozilla

N — Mozilla **J**

.net / java   .net / java

3, 1   1, 0   0, 0   2, 2

- (N, NN) is not a NEP when restricted to the subgame starting at **J**

- (J, JJ) is not a NEP when restricted to the subgame starting at **N**

- (N, NJ) is a subgame perfect Nash equilibrium

Mozilla

|		NN	NJ	JN	JJ
MS	N	3,1	3,1	1,0	1,0
	J	0,0	2,2	0,0	2,2

Subgame Perfect NEP

Not subgame Perfect NEP

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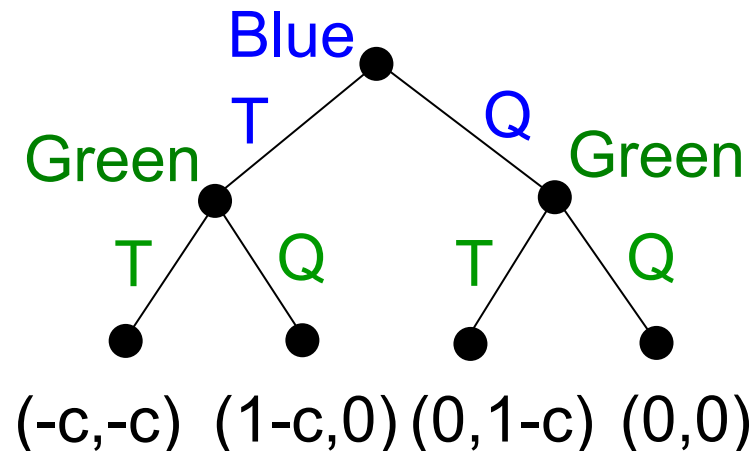   c. TDMA Transmission

Extensive-form Games of TDMA

Blue plays first, then Green plays.
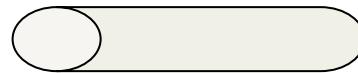
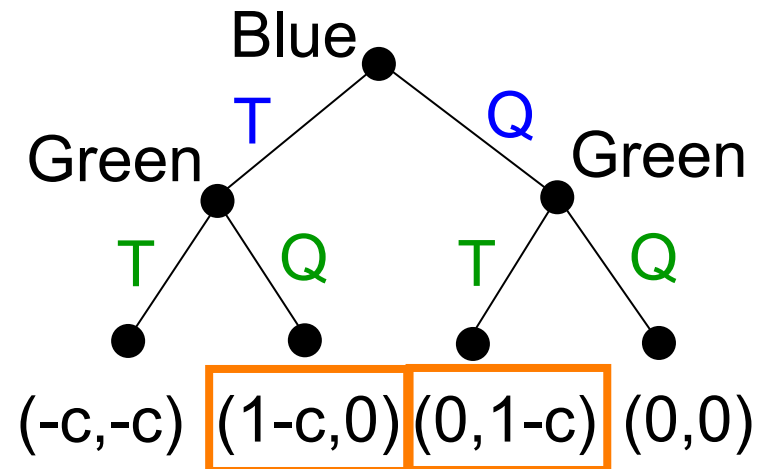Time-division channel

Reward for successful transmission: 1

Cost of transmission: c
(0 < c << 1)

Blue

T   Q

Green   Green

T   Q   T   Q

(-c,-c) (1-c,0) (0,1-c) (0,0)

Strategies in Dynamic Games

- strategies for Blue:

  T, Q

- strategies for Green:

  TT, TQ, QT and QQ

If they have to decide
independently: three Nash Equilibria

(T,QT), (T,QQ) and (Q,TT)

Blue

Green T Q Green

T Q T Q
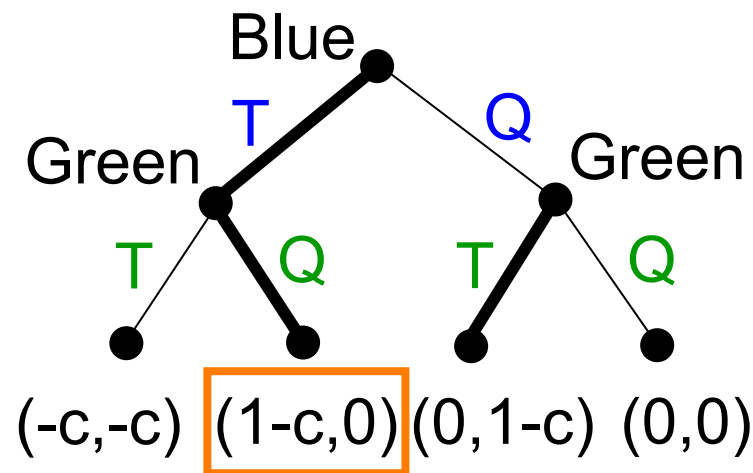
(-c,-c) (1-c,0) (0,1-c) (0,0)

Extensive to Normal Form

Green

Blue vs. Green	TT	TQ	QT	QQ
T	(-c,-c)	(-c,-c)	(1-c,0)	(1-c,0)
Q	(0,1-c)	(0,0)	(0,1-c)	(0,0)

Blue

Backward Induction

- Solve the game by reducing from the final stage

- Eliminates Nash equilibria that are *incredible threats*

incredible threat: (Q, TT)

Blue

T   Q

Green   Green

T   Q   T   Q

(-c,-c)   (1-c,0)   (0,1-c)   (0,0)

Sub-game Perfection

- Extends the notion of Nash equilibrium

One-deviation property: A strategy s_i conforms to the *one-deviation property* if there does not exist any node of the tree, in which a player i can gain by deviating from s_i and apply it otherwise.

Subgame perfect equilibrium: A strategy profile s constitutes a subgame perfect equilibrium if the one-deviation property holds for every strategy s_i in s.

Finding subgame perfect equilibria using backward induction

SPNE: (T, QT)

Blue

T   Q

Green   Green

T   Q   T   Q

(-c,-c) (1-c,0) (0,1-c) (0,0)