بسم الله الرحمن الرحيم

Foundations of Game Theory for Electrical and Computer Engineering

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1394

Let's play sequentially!

DYNAMIC GAMES I

Dynamic Games I

1. Sequential vs Simultaneous Moves
2. Extensive Forms (Trees)
3. Analyzing Dynamic Games: Backward Induction
4. Moral Hazard
5. Incentive Design
6. Norman Army vs. Saxon Army Game
7. Revisit Cournot Duopoly (Stackelberg Model)

Let's Play a Game!

"Cash in a Hat" Game

- Two players
- Player 1 strategies: put $0, $1 or $3 in a hat

- Then, the hat is passed to player 2

- Player 2 strategies: either **"match"** (i.e., add the same amount of money in the hat) or "**take**" the cash

"Cash in a Hat" Game

Payoffs:

- $U_1 = \begin{cases} \$0 \rightarrow \$0 \\ \$1 \rightarrow \text{if match net profit } \$1, -\$1 \text{ if not} \\ \$3 \rightarrow \text{if match net profit } \$3, -\$3 \text{ if not} \end{cases}$

- $U_2 = \begin{cases} \text{Match } \$1 \rightarrow \text{Net profit } \$1.5 \\ \text{Match } \$3 \rightarrow \text{Net profit } \$2 \\ \text{Take the cash} \rightarrow \$ \text{ in the hat} \end{cases}$

"Cash in a Hat" Game

- What would you do?

- How would you analyze this game?

- This game is a toy version of a more important game, involving a ***lender (Accel Partner)*** and a ***borrower (Facebook)***

Lender & Borrower Game

- The lender has to decide ***how much money to invest*** in the project

- After the money has been invested, the borrower could
  - Go forward with the project and work hard
  - Shirk, and run away with the money

Simultaneous vs. Sequential Moves

- Question: ***what is different about this game*** with regards to all the games we've played so far?
- This is a ***sequential move game***
  - What really makes this game a sequential move game?
  - It is not the fact that player 2 chooses after player 1, *so time is not the really key idea* here
  - The key idea is that player 2 can ***observe*** player 1's choice before having to make his or her choice
  - Notice: player 1 knows that this is going to be the case!

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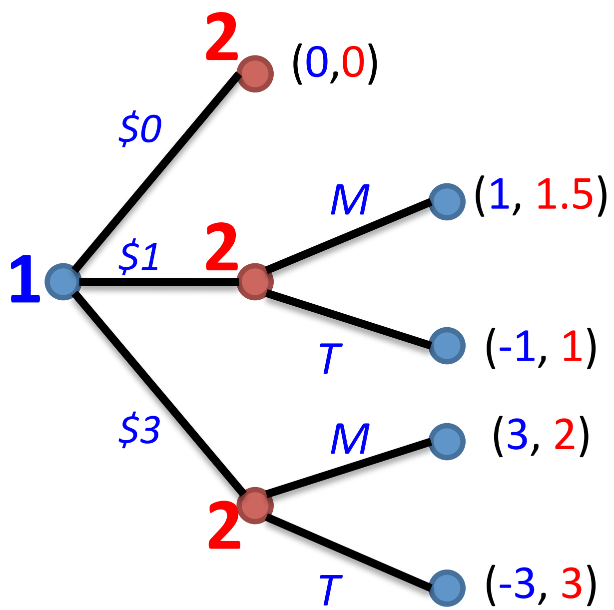
Extensive Form Game

- A useful representation of such games is ***game trees*** also known as the ***extensive form***
- For normal form games we used matrices, here we'll focus on trees
  - Each internal node of the tree will represent the ability of a player to make choices at a certain stage, and they are called ***decision nodes***
  - Leafs of the tree are called ***end nodes*** and represent payoffs to both players

"Cash in a hat" Representation



What do we do to analyze such game?

LET'S SOLVE THIS GAME!

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Analyzing Dynamic Games

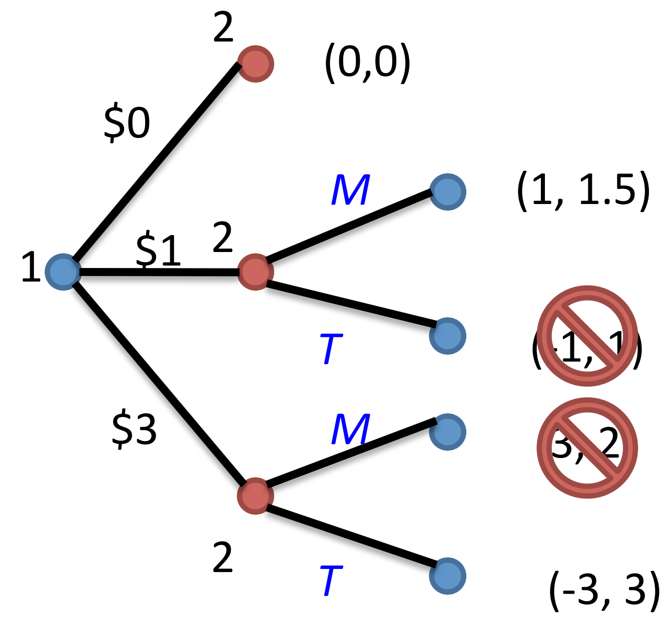
- Players that move early on in the game should ***put themselves in the shoes of other players***
- Here this reasoning takes the form of ***anticipation***
- Basically, look towards the end of the tree and work back your way along the tree to the root

Backward Induction

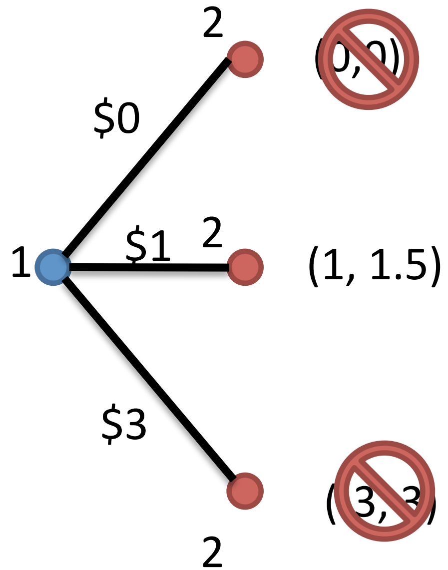
- Start with the last player and chose the strategies yielding higher payoff
- This simplifies the tree
- Continue with the before-last player and do the same thing
- Repeat until you get to the root

This is a fundamental concept in game theory

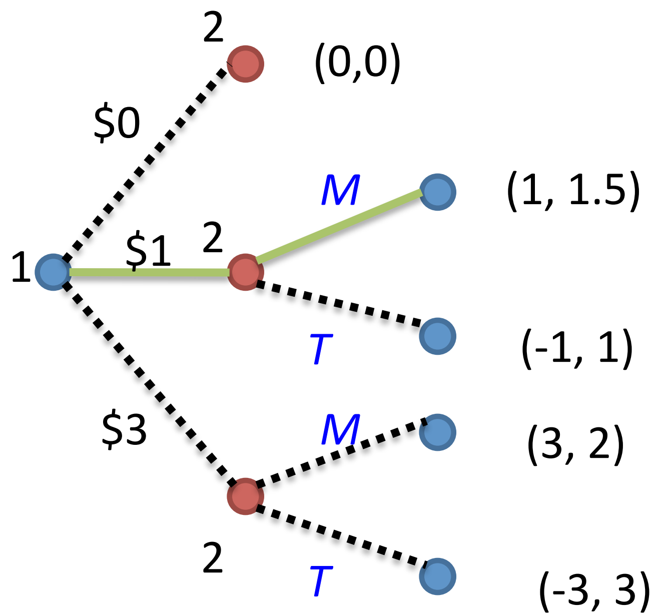
Backward Induction



Backward Induction



Backward Induction



Player 1 chooses to invest $1, Player 2 matches

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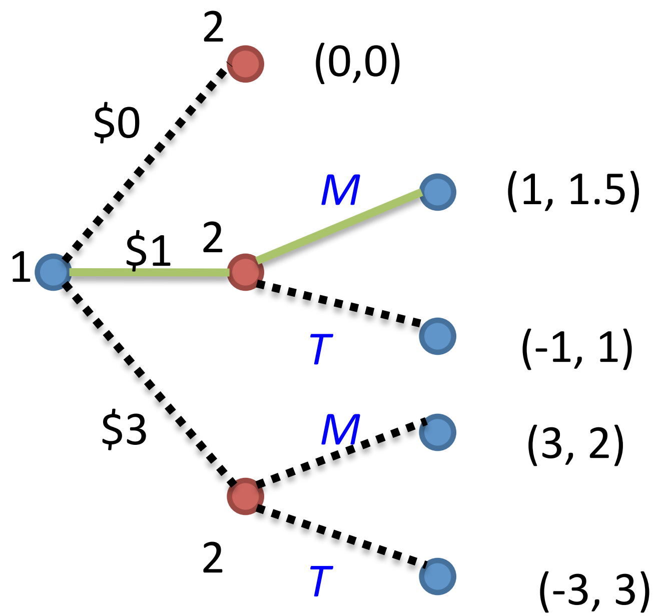
What is the problem in the outcome of this game?



Very similar to what we learned with the Prisoners' Dilemma

The Problem!

- It is not a disaster:
  - The lender doubled her money
  - The borrower was able to go ahead with a small scale project and make some money
- But, we would have liked to end up in another branch:
  - Larger project funded with $3 and an outcome better for both the lender and the borrower
- What does prevent us from getting to this latter good outcome?

Moral Hazard

One player (the borrower) has incentives to do things that are not in the interests of the other player (the lender)

– By giving a **too big loan**, the incentives for the borrower will be such that they will not be aligned with the incentives on the lender

– Notice that moral hazard has also **disadvantages for the borrower**

Moral Hazard: an Example

- Insurance companies offers "full-risk" policies
- People subscribing for this policies may have no incentives to take care!

- In practice, insurance companies force me to bear some deductible costs ("franchise")

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How can we solve the Moral Hazard problem?

- We've already seen one way of solving the problem ➔ keep your project small

- Are there any other ways?

Introduce Laws

- Similarly to what we discussed for the PD
- Example: ***bankruptcy laws***
- But, there are limits to the degree to which borrowers can be punished

- The borrower can say: I can't repay, I'm bankrupt
- And he/she's more or less allowed to have a fresh start

Limits/Restrictions on Money

- Another way could be to asking the borrowers a concrete plan (***business plan***) on how he/she will spend the money
- This boils down to changing the order of play!

- But, what's the problem here?
- Lack of flexibility, which is the motivation to be an entrepreneur in the first place!
- Problem of timing: it is sometimes hard to predict up-front all the expenses of a project

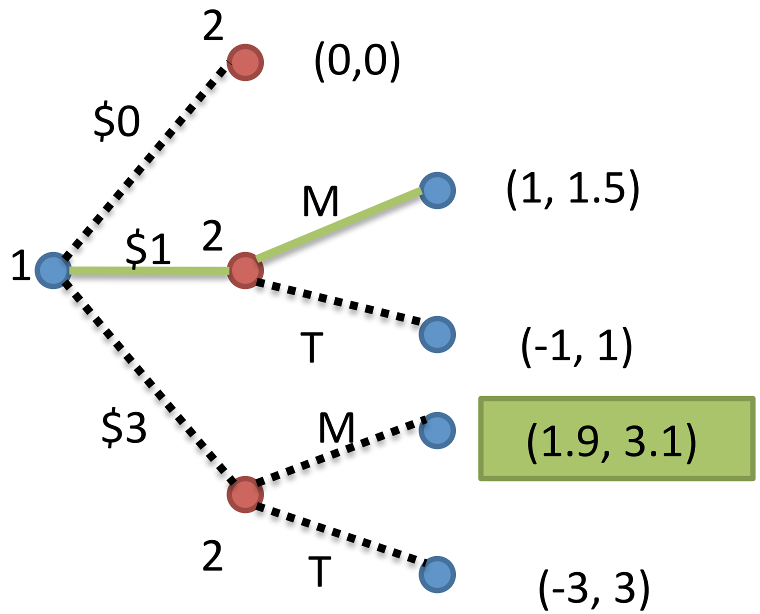
Break the Loan Up

- Let the loan come in small installments
- If a borrower does well on the first installment, the lender will give a bigger installment next time

- It is similar to taking this one-shot game and turn it into a ***<u>repeated game</u>***
  – We will see similar thing in the PD game

Change Contract to Avoid Shirk

- The borrower could re-design the payoffs of the game in case the project is successful



Incentive Design (1)

- Incentives have to be designed when defining the game in order to achieve goals
- Notice that in the last example, the lender is not getting a 100% their money back, but they end up doing better than what they did with a smaller loan

Sometimes a smaller share of a larger pie can be bigger than a larger share of a smaller pie

Incentive Design (2)

- In the example we saw, even if \$1.9 is larger than \$1 in ***absolute terms***, we could look at a different metric to judge a lenders' actions

- ***Return on Investment*** (ROI)
  - For example, as an investment banker, you could also just decide to invest in 3 small projects and get 100% ROI

Incentive Design (3)

- So should an investment bank care more about absolute payoffs or ROI?

- It depends! On what?

Incentive Design (4)

- There are two things to worry about:
  - The funds supply
  - The demand for your cash (the project supply)

- If there are few projects you may want to maximize the absolute payoff
- If there are infinite projects you may want to maximize your ROI

Incentive Design Per-Se!

1. Peer-to-Peer Networking
2. Mobile/Grid/Cloud Computing
3. Privacy and Security
4. Cooperation Designs

We won’t go into the details in this lecture!

Beyond Incentives…

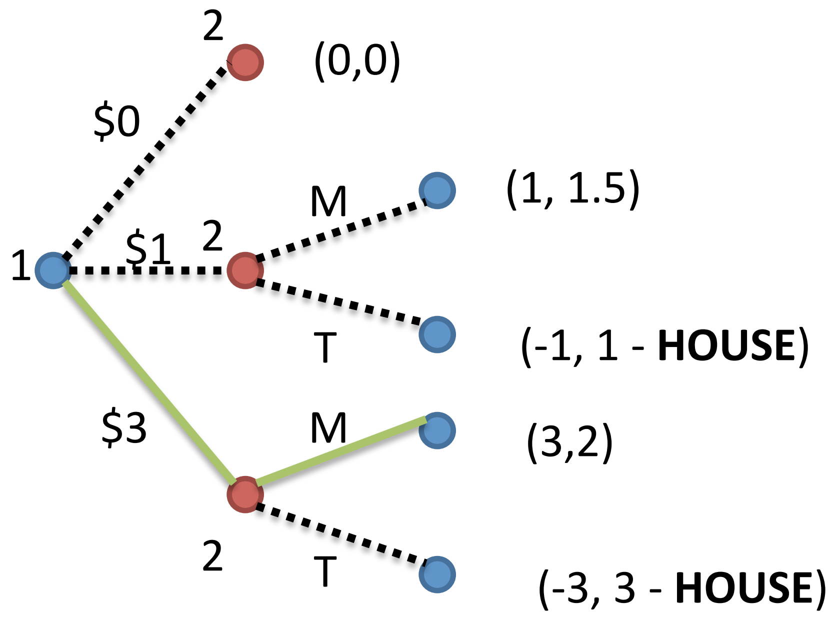
- Can we do any other things rather than providing incentives?

- Ever heard of "***collateral***"?
  - Example: subtract house from run away payoffs

  ➔ Lowers the payoffs to borrower at some tree points, yet makes the borrower better off!

Collateral example

- The borrower could re-design the payoffs of the game in case the project is successful



Collaterals

- They do hurt a player enough to change his/her behavior

→**Lowering the payoffs** at certain points of the game, **does not mean** that a player will be worse off!!

- Collaterals are part of a larger branch called ***commitment strategies***
  - Next, an example of commitment strategies

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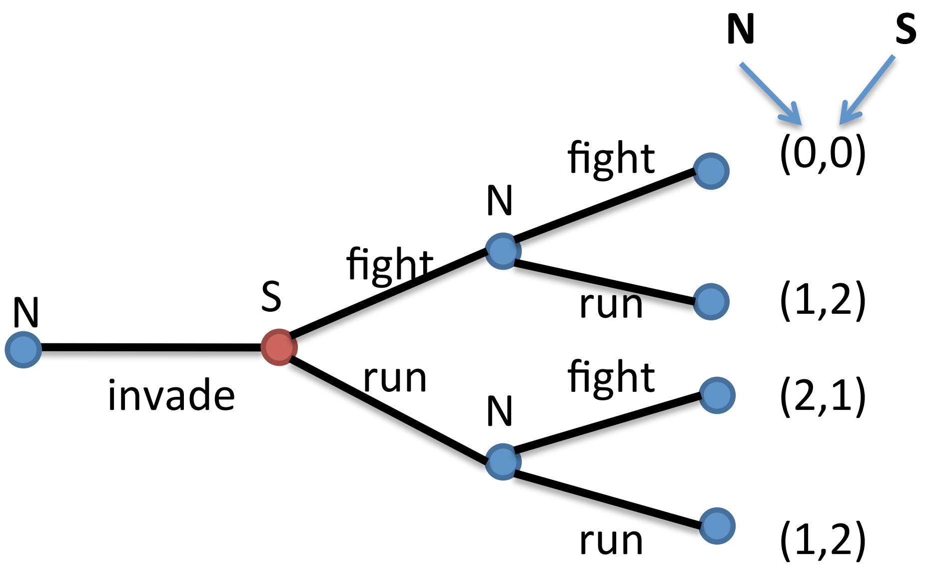
Norman Army vs. Saxon Army Game

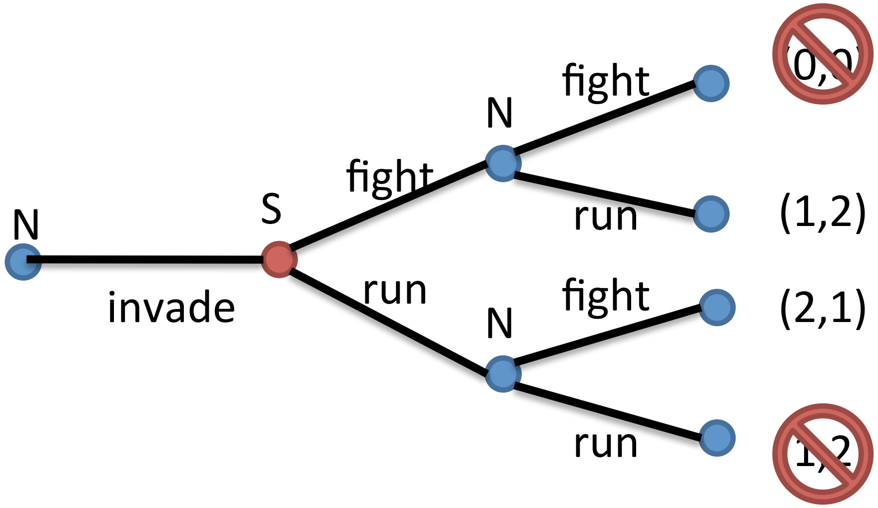
- Back in 1066, William the Conqueror led an invasion from Normandy on the Sussex beaches
- We're talking about ***military strategy***
- So basically we have two players (the armies) and the strategies available to the players are whether to "fight" or "run"

Norman Army vs. Saxon Army Game

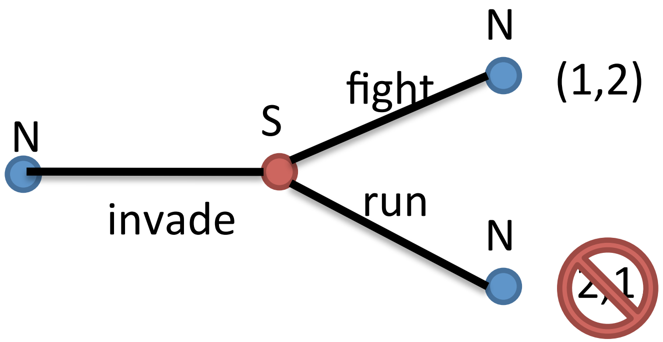


Let's analyze the game with
Backward Induction

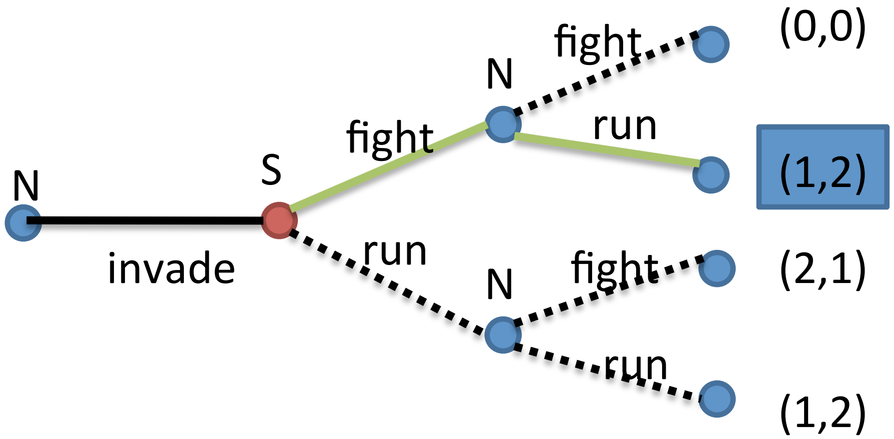
Norman Army vs. Saxon Army Game



Norman Army vs. Saxon Army Game



Norman Army vs. Saxon Army Game

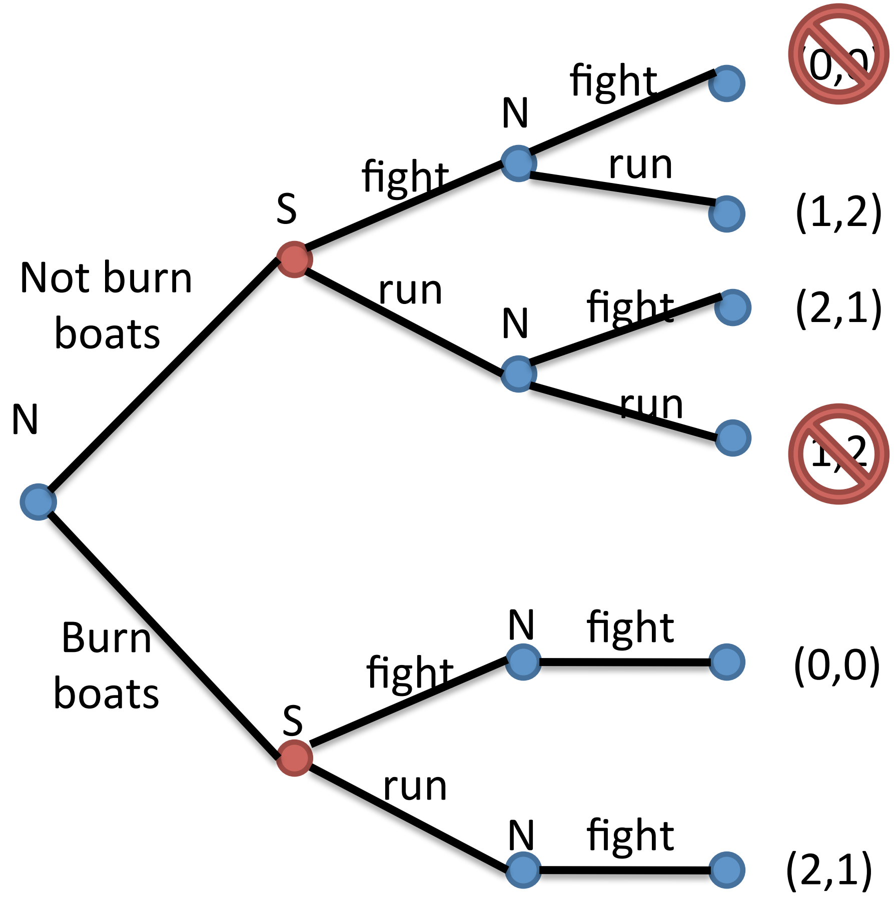


Backward Induction tells us:

- Saxons will fight
- Normans will run away

→ What did William the Conqueror did?

Norman Army vs. Saxon Army Game



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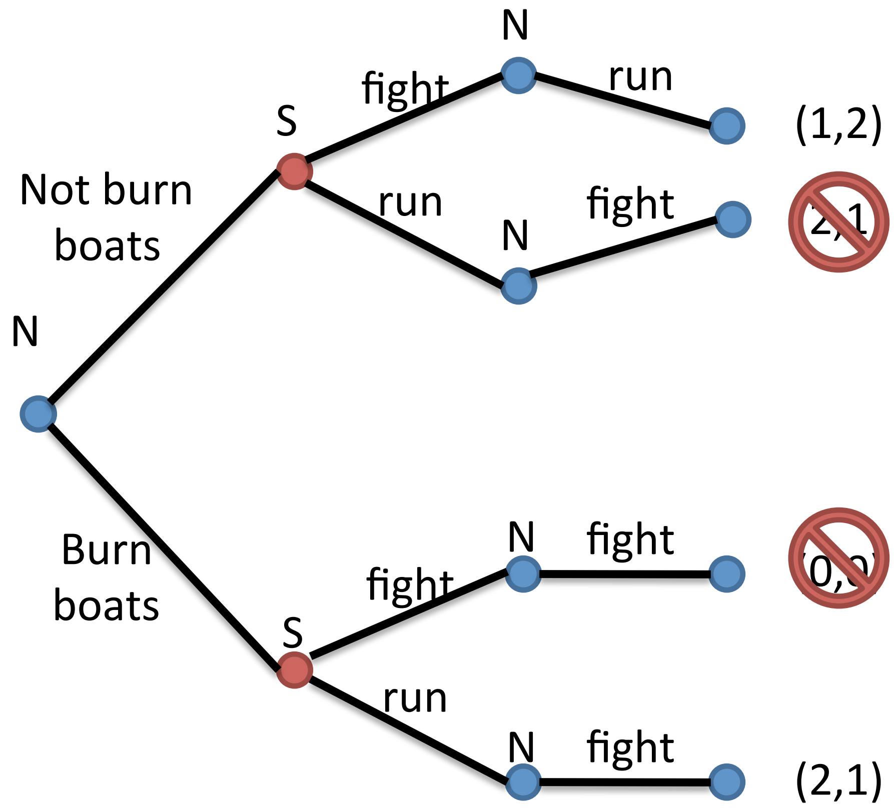

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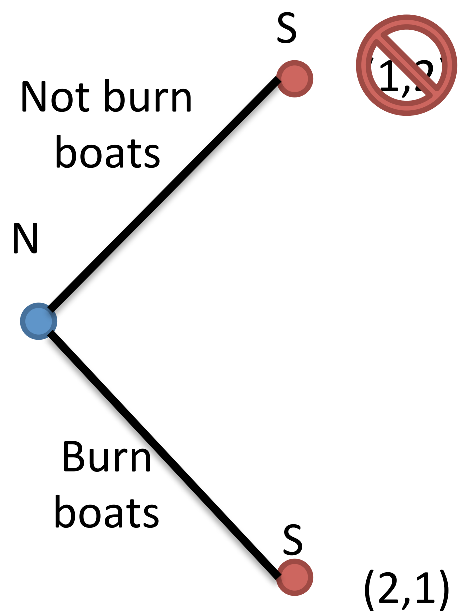

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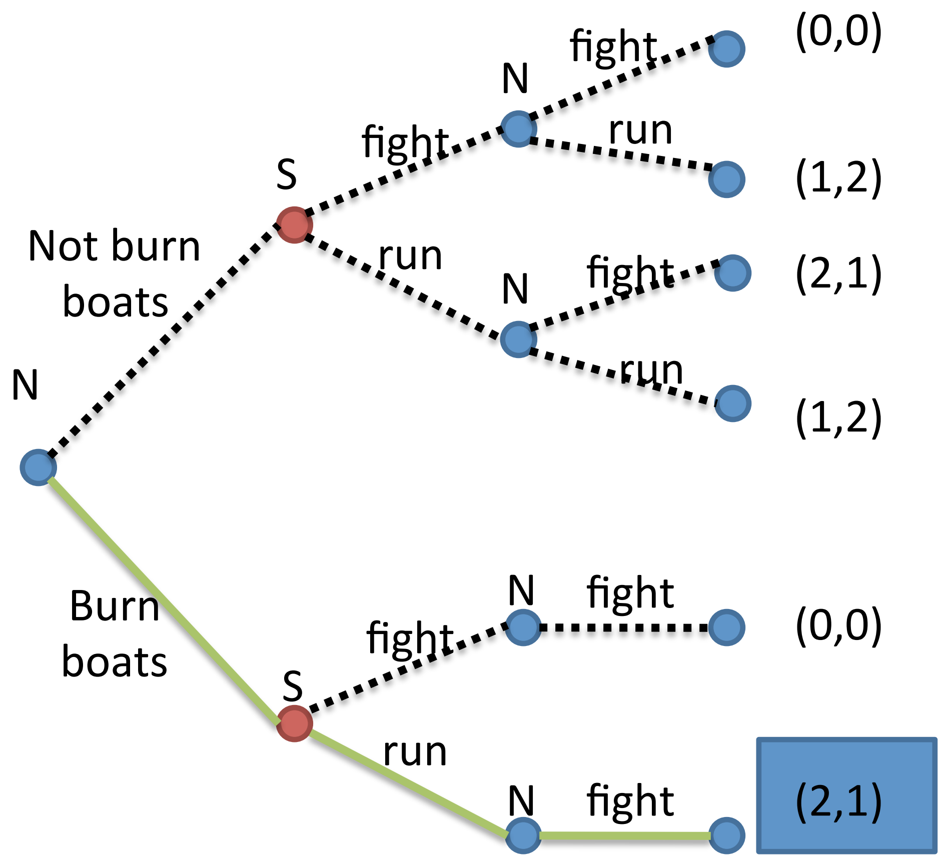

Lesson learned

- Sometimes, getting rid of choices can make me better off!

- ***Commitment***:
  - Fewer options change the behavior of others

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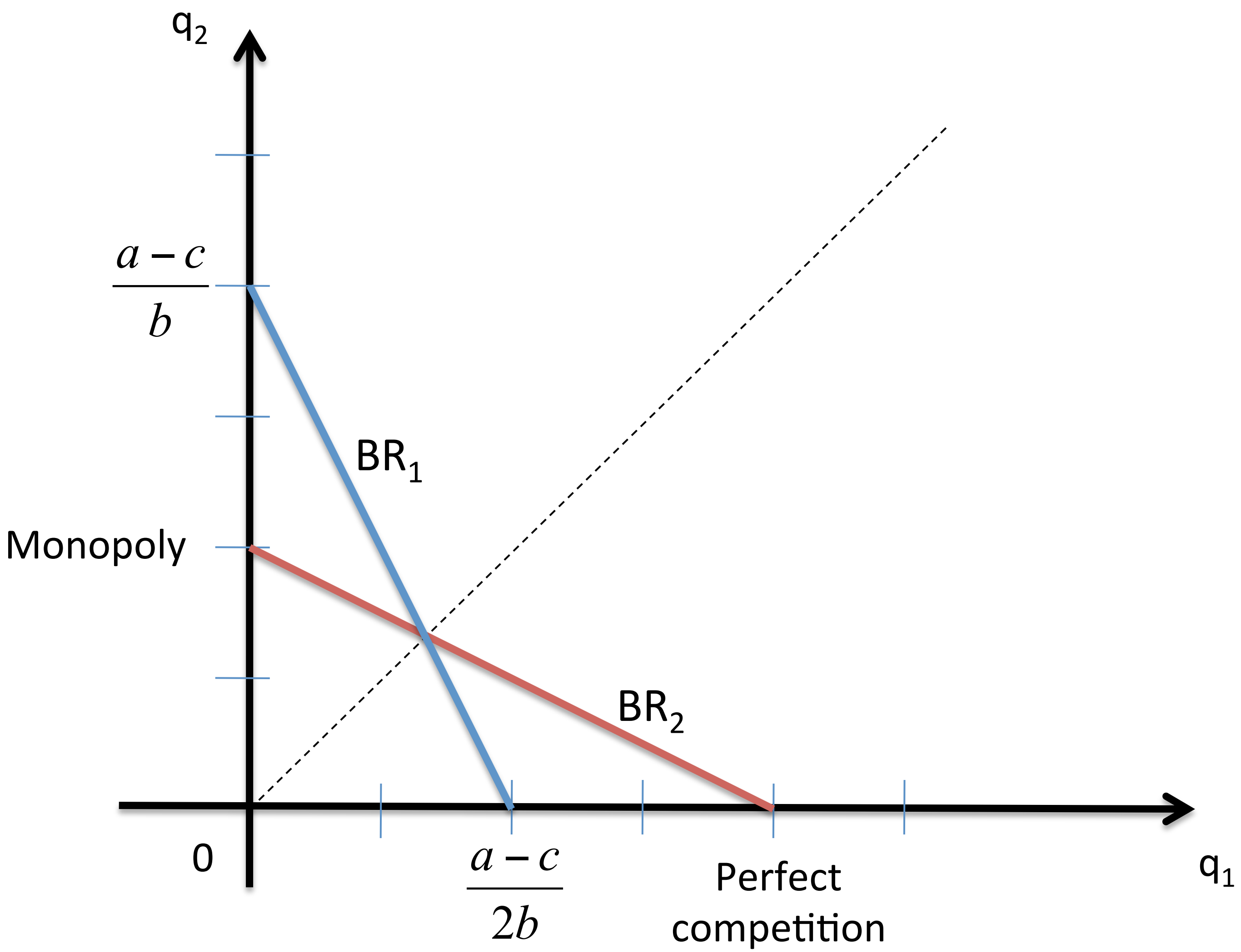
REVISITING COURNOT DUOPOLY



The game is symmetric

What is the NE of the Cournot Duopoly?

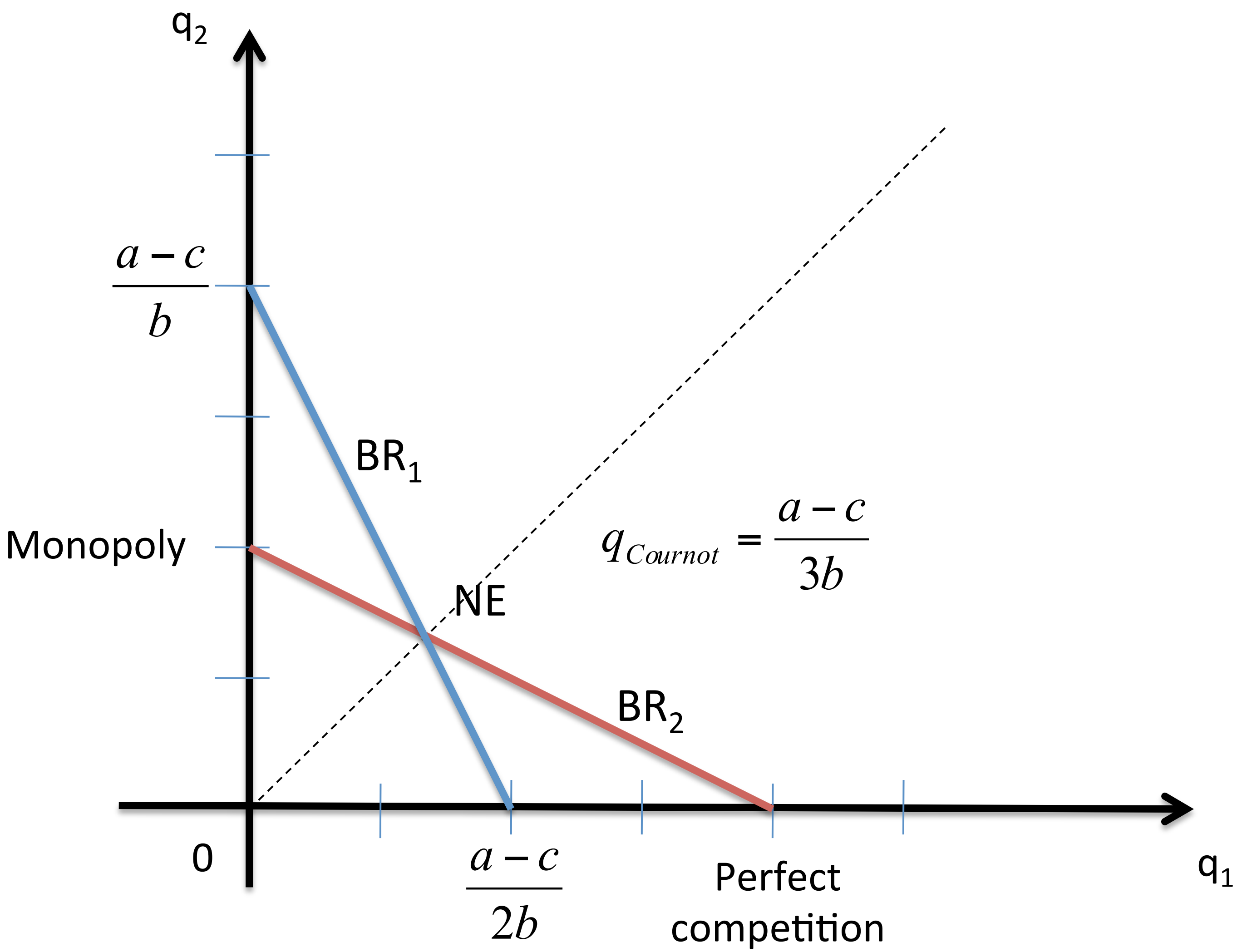
- Graphically we've seen it, formally we have:

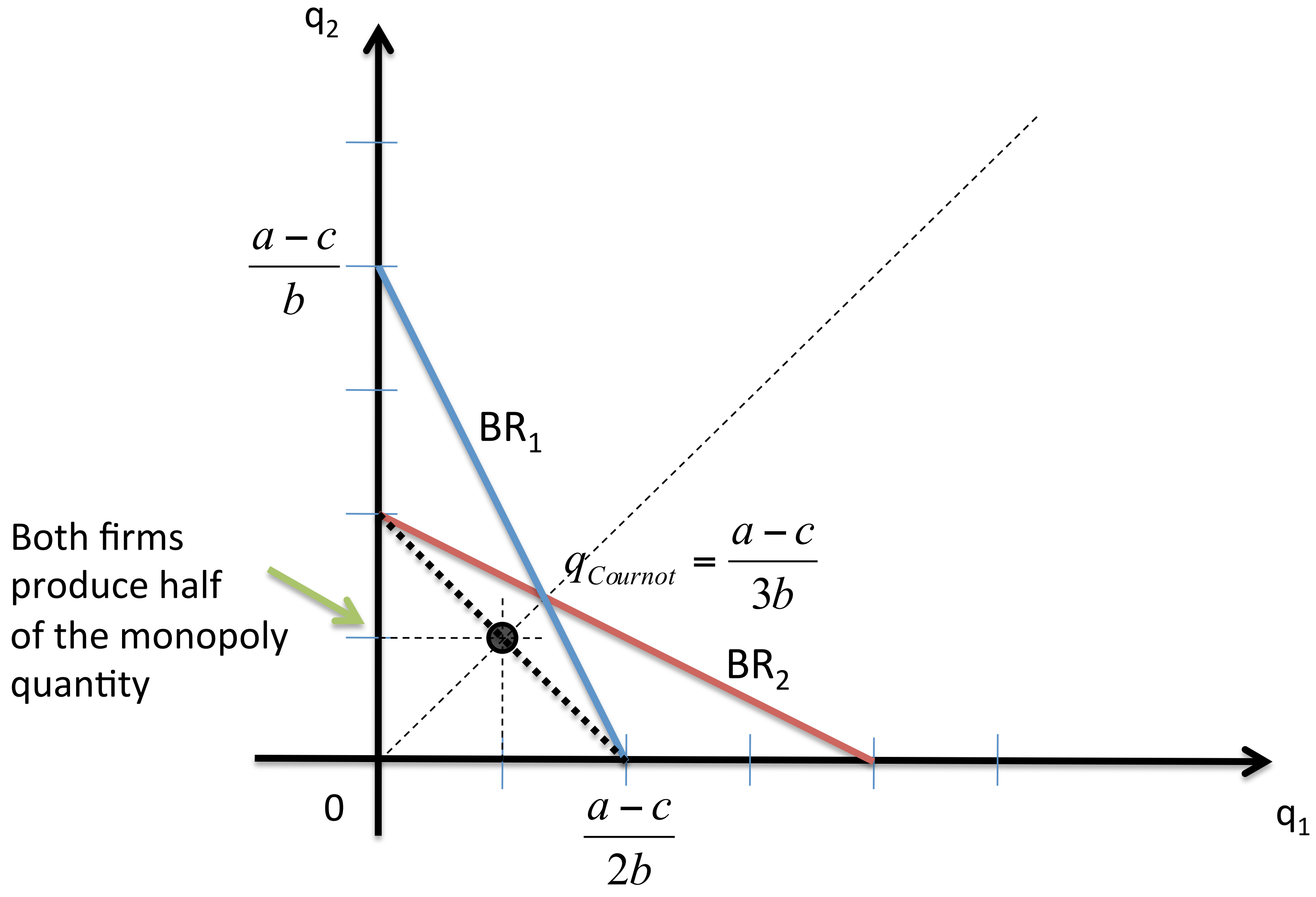
$$BR_1(q_2) = BR_2(q_1) \Rightarrow q_1^* = q_2^*$$

$$\frac{a-c}{2b} - \frac{\hat{q}_2}{2} = \hat{q}_2$$

$$\Rightarrow q_1^* = q_2^* = \frac{a-c}{3b}$$

- We have found the ***COURNOT QUANTITY***


q2
a − c / b
Monopoly
BR1
NE
q_Cournot = (a − c) / 3b
BR2
0
a − c / 2b
Perfect competition
q1



q_2
$\frac{a-c}{b}$
BR_1
Both firms produce half of the monopoly quantity
$q_{Cournot} = \frac{a-c}{3b}$
BR_2
0
$\frac{a-c}{2b}$
q_1

Stackelberg Model

- We are going to assume that one firm gets to move first and the other moves after
  - That is one firm gets to set the quantity first
- Assuming we're in the world of competition, ***is it an advantage to move first***?
  - Or maybe it is better to wait and see what the other firm is doing and then react?

- We are going to use ***backward induction***

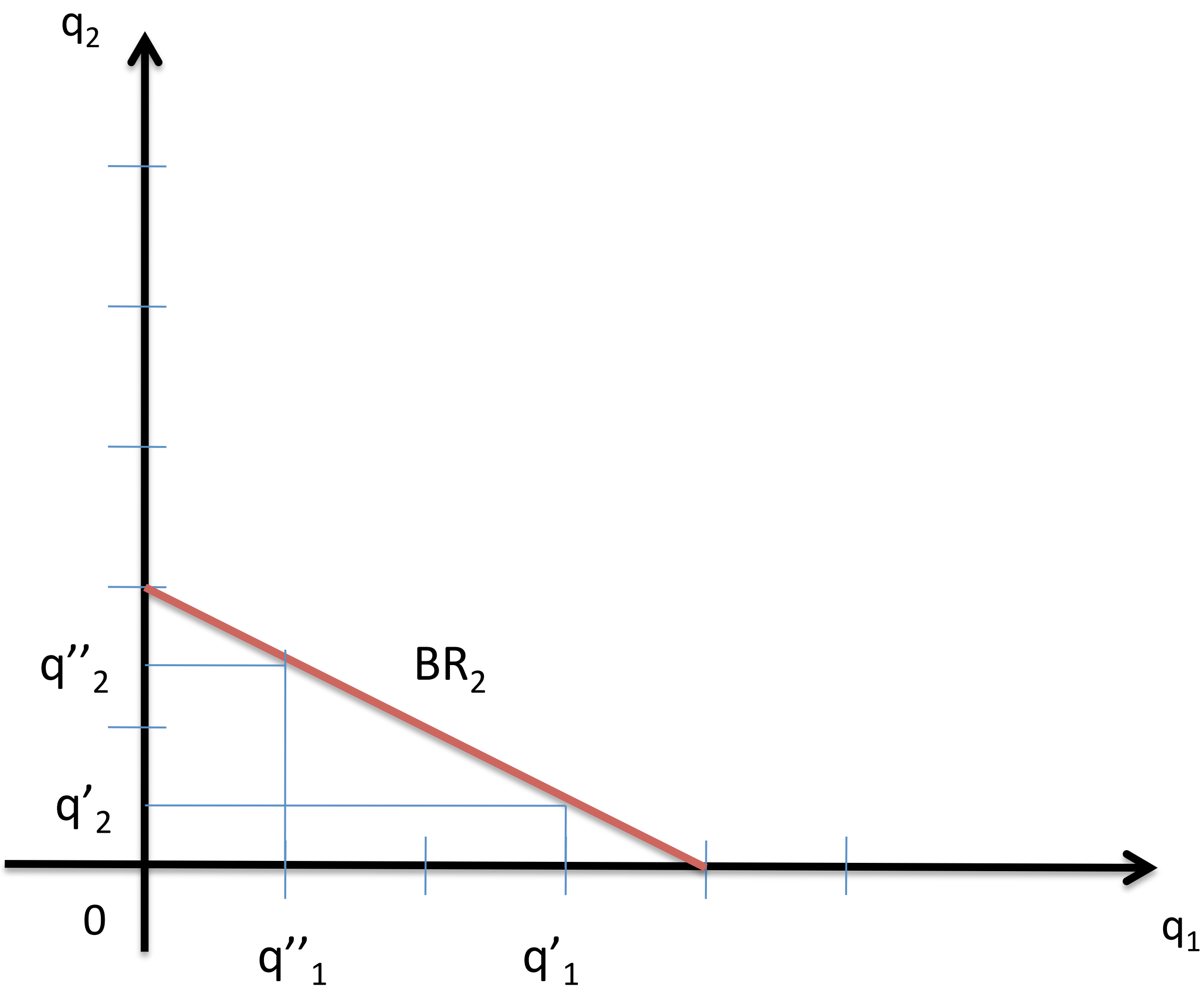
Stackelberg Model

- Unfortunately we won't be able to draw trees, as the game is too complex

- First we'll go for an intuitive explanation of what happens, then we'll figure out the math

Stackelberg Model

- Let's assume firm 1 moves first
- Firm 2 is going to observe firm 1's choice and then move
- How would you go about it?


q2
q''2
BR2
q'2
0
q''1
q'1
q1

Stackelberg Model

- By definition of Best Response, we know what's the profit maximizing strategy of firm 2, given an output quantity produced by firm 1

- Now we know what firm 2 will do, what's interesting is to look at what firm 1 will come up with

Stackelberg Model

- What quantity should firm 1 produce, knowing that firm 2 will respond using the BR?
  - This is a ***constrained optimization problem***

- One **legitimate question** would be: should firm 1 produce more or less than the quantity she produced when the moves were simultaneous?
  - In particular, should firm 1 produce more or less than the Cournot quantity?

Stackelberg Model

- Question: should firm 1 produce more than

$$q_1^* = \frac{a-c}{3b}$$

- Remember, we are in a ***strategic substitutes*** setting
  - The more firm 1 produces, the less firm 2 will produce and vice-versa
- Firm 1 producing more ➔ firm 1 is happy

Stackelberg Model

- If q_1 increases, then q_2 will decrease (as suggested by the BR curve)
- What happens to firm 1's profits?
  - They go up, for otherwise firm 1 wouldn't have set higher production quantities
- What happens to firm 2's profits?
  - The answer is not immediate
- What happened to the total output in the market?
  - Even here the answer is not immediate

Stackelberg Model

- Let's have a nerdy look at the problem:

$$p = a - b(q_1 + q_2)$$

$$profit_i = pq_i - cq_i$$

- Let's apply the Backward Induction principle
  - First, solve the maximization problem for firm 2, taking q_1 as given
  - Then, focus on firm 1

Stackelberg Model

- Let's focus on firm 2:

$$\max_{q_2}\left[\left(a-bq_1-bq_2\right)q_2-cq_2\right]$$

$$\frac{\partial}{\partial q_2}\Rightarrow q_2=\frac{a-c}{2b}-\frac{q_1}{2}$$

- We now can take this quantity and plug it in the maximization problem for firm 1

Stackelberg Model

- Let's focus on firm 1:

$$\max_{q_1}\left[\left(a - bq_1 - bq_2\right)q_1 - cq_1\right] =$$

$$\max_{q_1}\left[\left(a - bq_1 - b\left(\frac{a-c}{2b} - \frac{q_1}{2}\right)\right) - c\right]q_1 =$$

$$\max_{q_1}\left[\frac{a-c}{2} - \frac{bq_1}{2}\right]q_1 = \max_{q_1}\left[\frac{a-c}{2}q_1 - b\frac{q_1^2}{2}\right]$$

Stackelberg Model

- Let's derive F.O.C. and S.O.C.

$$\frac{\partial}{\partial q_1} = 0 \Rightarrow \frac{a-c}{2} - bq_1 = 0$$

$$\frac{\partial^2}{\partial q_1^2} = -b < 0$$

Stackelberg Model

- This gives us:

$$q_1 = \frac{a-c}{2b}$$

$$q_2 = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{2b} = \frac{a-c}{4b}$$

$$q_1^{NEW} > q_1^{Cournot}$$

$$q_2^{NEW} < q_2^{Cournot}$$

Stackelberg Model

- All this math to verify our initial intuition!

$$q_1^{NEW} > q_1^{Cournot}$$

$$q_2^{NEW} < q_2^{Cournot}$$

$$q_1^{NEW} + q_2^{NEW} = \frac{3(a-c)}{4b} > \frac{2(a-c)}{3b} = cournot$$

Observations

- Is what we've looked at really a sequential game?
- Despite we said firm 1 was going to move first, there's no reason to assume she's really going to do so!
- What do we miss?

Observations

- We need a ***commitment***
- In this example, sunk cost could help in believing firm 1 will actually play first

→ Assume firm 1 was going to invest a lot of money in building a plant to support a large production: this would be a credible commitment!

Observations

- Let's make an example: assume the two firms are "X" and "Y" trying to gain market shares for Z production in a city

- Suppose there's a board meeting where the strategy of the firms are decided

- What could Y do to deviate from Cournot?

Observations

- An example would be to be somehow "dishonest" and hire a spy to gain more information on X's strategy!

- To make the scenario even more intriguing, let's assume X knows that there's a spy in the board room
  - What should X do?

Simultaneous vs. Sequential

- There are some key ideas involved here

1. Games being simultaneous or sequential is ***not really about timing it is about information***
2. Sometimes, **more information can hurt**!
3. Sometimes, **more options can hurt**!