بسم الله الرحمن الرحیم

# Foundations of Game Theory for Electrical and Computer Engineering


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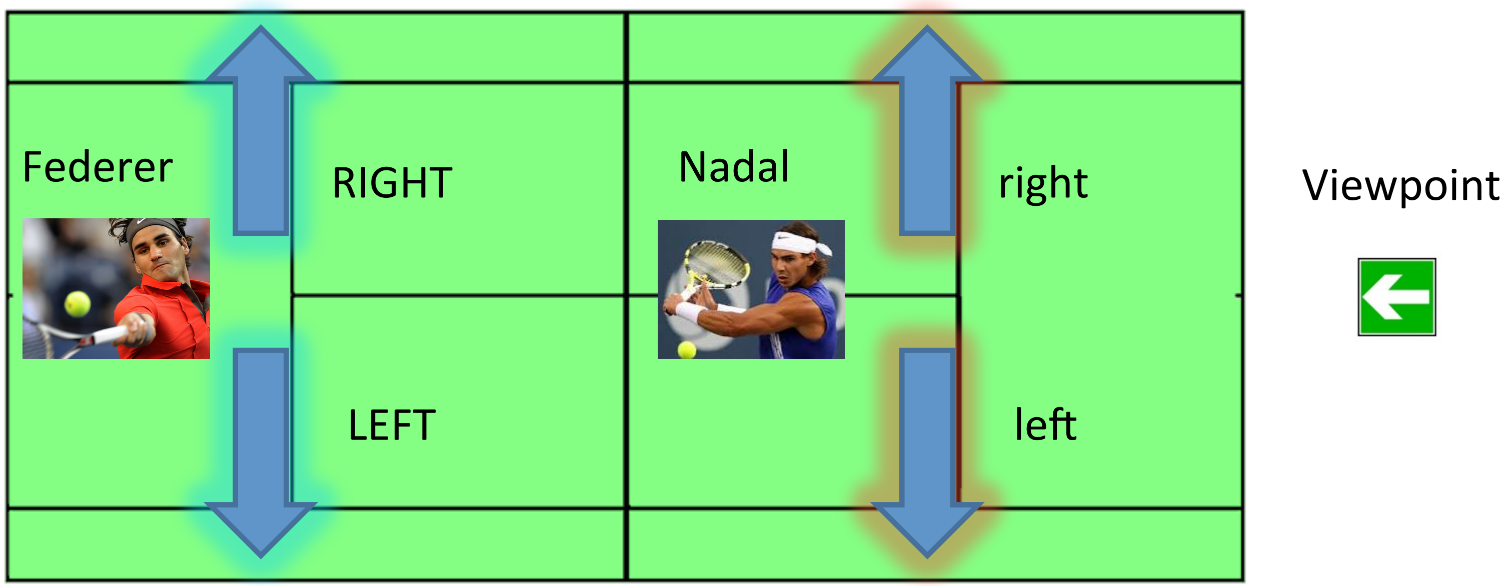
# MIXED STRATEGY NASH EQUILIBRIUM: AN EXAMPLE

# Tennis Game

- We're going to look at a tennis game
- Assume two players (Federer and Nadal)
- Where Nadal is at the net



# Tennis Game

| | | Nadal *l* | Nadal *r* | |
|---|---|---|---|---|
| Federer | ***L*** | 50,50 | 80,20 | p |
| | ***R*** | 90,10 | 20,80 | (1-p) |
| | | q | (1-q) | |

- Have a look at the payoffs
  - E.g.: if Federer chooses 'L' and Nadal guesses wrong and jumps to the 'r', Federer wins the point 80% of the time
- Is there any dominated strategy?
- Is there a ***pure strategy*** NE profile?

# Tennis Game

Let's find the mixed strategy NE

***Lesson 1***: Each player's randomization is the best response to the other player's randomization

***Lesson 2***: If players are playing a mixed strategy as part of a NE, then each of the pure strategies involved in the mix must itself be a best response

# Tennis Game

- Find a mixture for Nadal and one for Federer that are in equilibrium

- ***<u>TRICK</u>***:
  - To find Nadal's mix ($q$) I'm going to ***put myself in Federer's shoes*** and look at his payoffs
  - And vice-versa for Federer's mix ($p$)

# Tennis Game

- Federer's expected payoffs:

$$E\left[U_{Federer}\left(L,(q,1-q)\right)\right] = 50q + 80(1-q)$$

$$E\left[U_{Federer}\left(R,(q,1-q)\right)\right] = 90q + 20(1-q)$$

- If Federer is mixing in this NE then the payoff to the left and to the right must be equal, they must both be best responses
  - Otherwise Federer would not be mixing

# Tennis Game

- Federer's expected payoffs must be equal:

$$E\left[U_{Federer}\left(L,\left(q,1-q\right)\right)\right]=50q+80(1-q)$$

$$E\left[U_{Federer}\left(R,\left(q,1-q\right)\right)\right]=90q+20(1-q)$$

$$\Rightarrow 50q+80(1-q)=90q+20(1-q)$$

$$\Rightarrow 40q=60(1-q)$$

$$\Rightarrow q=0.6$$

- I was able to derive ***Nadal's mixing probability***
- This is the solution to the equation in one unknown that equates Federer's payoffs in the mix

# Tennis Game

- Nadal's expected payoffs:

$$E\left[U_{Nadal}\left(\left(p,1-p\right),l\right)\right]=50p+10(1-p)$$

$$E\left[U_{Nadal}\left(\left(p,1-p\right),r\right)\right]=20p+80(1-p)$$

$$\Rightarrow 50p+10(1-p)=20p+80(1-p)$$

$$\Rightarrow 30p=70(1-p)$$

$$\Rightarrow p=0.7$$

| | *l* | *r* | |
|---|---|---|---|
| ***L*** | 50,50 | 80,20 | p |
| ***R*** | 90,10 | 20,80 | (1-p) |
| | q | (1-q) | |

- Similarly, we computed Federer's mixing probability

# Tennis Game

- We found the mixed strategy NE:

**Federer Nadal**

➔ [(0.7, 0.3) , (0.6, 0.4)]

**L R l r**

- What would happen if Nadal jumped to the left more often than 0.6?
  – Federer would be better of **playing the pure strategy 'R'!**
- What if he jumped less often than 0.6?
  – Federer would be **shooting to the 'L' all time!**

# Tennis Game

| | | Nadal *l* | Nadal *r* | |
|---|---|---|---|---|
| Federer | ***L*** | 30,70 | 80,20 | p |
| | ***R*** | 90,10 | 20,80 | (1-p) |
| | | q | (1-q) | |

- Suppose a new coach teaches Nadal how to forehand, and the payoff would change accordingly
- There is still no pure strategy NE
- What would happen in this game?

# Tennis Game

- Let's first let our intuition work
- Basically Nadal is better at his forehand and when Federer shoots there, Nadal scores more often than before

➔ ***Direct effect***: Nadal should increase his $q$

- But, Federer knows Nadal is better at his forehand, hence he will shoot there less often

➔ ***Indirect effect***: Nadal should decrease his $q$

# Tennis Game

- Let's compute again q:

$$E\left[U_{Federer}\left(L,\left(q,1-q\right)\right)\right]=30q+80(1-q)$$

$$E\left[U_{Federer}\left(R,\left(q,1-q\right)\right)\right]=90q+20(1-q)$$

$$\Rightarrow 30q+80(1-q)=90q+20(1-q)$$

$$\Rightarrow 60q=60(1-q)$$

$$\Rightarrow q=0.5$$

- We see that in the end Nadal's q went down from 0.6 to 0.5!!
- **The indirect effect was predominant**

# Tennis Game

- Nadal's expected payoffs:

$$E\left[U_{Nadal}\left(\left(p,1-p\right),l\right)\right]=70p+10(1-p)$$

$$E\left[U_{Nadal}\left(\left(p,1-p\right),r\right)\right]=20p+80(1-p)$$

$$\Rightarrow 50p+10(1-p)=20p+80(1-p)$$

$$\Rightarrow 50p=70(1-p)$$

$$\Rightarrow p=7/12=0.5833<0.7$$

- **The direct effect was predominant**
- Federer will be shooting to the left with less probability

# Tennis Game: Summary

- We just performed a ***comparative statistics*** exercise
  - We looked at a game and found an equilibrium, then we perturbed the original game and found another equilibrium and compared the two NE
- Suppose Nadal's $q$ had not changed
  - Federer would have never shot to the left
  - But this couldn't be a mixed strategy NE
  - There was a force to put back things at equilibrium and that was the force that pulled down Nadal's $q$

# NASH THEOREM

Every Finite Game has
a Mixed Strategy Nash Equilibrium

- Why is this important?
- Without knowing the existence of an equilibrium, it is difficult (perhaps meaningless) to try to understand its properties.
- Armed with this theorem, we also know that every finite game has an equilibrium, and thus we can simply try to locate the equilibria.

# LET'S SEE PAYOFFS!

# Tennis Game

| | | **Nadal** | | |
|---|---|---|---|---|
| | | ***l*** | ***r*** | |
| **Federer** | ***L*** | 50,50 | 80,20 | p |
| | ***R*** | 90,10 | 20,80 | (1-p) |
| | | q | (1-q) | |

- We identified the ***mixed strategy NE*** for this game

Federer Nadal

➔[(0.7, 0.3) , (0.6, 0.4)]

L R l r

p* (1-p*) q* (1-q*)

# Tennis Game

- How do we actually ***check*** that this is indeed an equilibrium?
- Let's verify that in fact p* is BR(q*)
- Federer's payoffs:
  - Pure strategy L ➔ 50*0.6 + 80*0.4 = 62
  - Pure strategy R ➔ 90*0.6 + 20*0.4 = 62
  - Mix p* ➔ 0.7*62 + 0.3*62 = 62
- Nadal's payoffs:
  - Pure Strategy l ➔ 50*0.7 + 10*0.3 = 38
  - Pure Strategy r ➔ 20*0.7 + 80*0.3 = 38
  - Mix q* ➔ 0.6*38 + 0.4*38 = 38

- Federer has no ***strictly profitable*** ***pure-strategy*** deviation

**Note (again): You cannot always win by playing NE**

# Tennis Game

- But is this enough? There are no pure-strategy deviations, but could there be any other mixes?
- Any mixed strategy yields a payoff that is a weighted average of the pure strategy payoffs
  - This already tells us: if you didn't find any pure-strategy deviations then you'll not find any other mixes that will be profitable

To check if a mixed strategy is a NE
we only have to check if
there are any pure-strategy profitable deviations

# Discussion

- Since we're in a mixed strategy equilibrium, it must be the case that the payoffs are equal
- Indeed, if it was not the case, then you shouldn't be randomizing!!

# Applied Example: Security Check at Airport

- After the security problems in the U.S. and worldwide airports due to high risks of attacks, the need for devices capable of inspecting luggage has raised considerably

- The problem is that there are not enough of such machines

- Wrong statements have been promoted by local governments:
  - If we put a check device in NY then all attacks will be shifted to Boston, but if we put a check device in Boston, the attacks will be shifted to yet another city

  ➔The claim was that whatever the security countermeasure, it would only shift the problem

# Applied Example: Security Check at Airport

What if you **wouldn't notify** where you would actually put the check devices, which boils down to randomizing?

The hard thing to do in practice is how to mimic randomization!!

# MIXED STRATEGIES NE: INTERPRETATIONS

# The Battle of the Sexes

| | | Player 2 | | |
|---|---|---|---|---|
| | | M | N | |
| Player 1 | M | 2,1 | 0,0 | p |
| | N | 0,0 | 1,2 | (1-p) |
| | | q | 1-q | |

- We already know a lot about this game
- There are two pure-strategy NE:
(M,M) and (N,N)
- We know that there is a problem of ***coordination***
- We know that without communication, it is possible (and quite probable) that the two players might fail to coordinate

# The Battle of the Sexes

- Player 1 perspective, find NE q:

$$\left.\begin{aligned} E\left[U_1\left(M,(q,1-q)\right)\right] &= 2q + 0(1-q) \\ E\left[U_1\left(N,(q,1-q)\right)\right] &= 0q + 1(1-q) \end{aligned}\right\} 2q = (1-q) \Rightarrow q = \frac{1}{3}$$

- Player 2 perspective, find NE p:

$$\left.\begin{aligned} E\left[U_2\left((p,1-p),M\right)\right] &= 1p + 0(1-p) \\ E\left[U_2\left((p,1-p),N\right)\right] &= 0p + 2(1-p) \end{aligned}\right\} 1p = 2(1-p) \Rightarrow p = \frac{2}{3}$$

# The Battle of the Sexes

- Let's check that p=2/3 is indeed a BR for Player1:

$$\left.\begin{aligned} E\left[U_1\left(M,\left(\frac{1}{3},\frac{2}{3}\right)\right)\right] &= 2\frac{1}{3}+0\frac{2}{3} \\ E\left[U_1\left(N,\left(\frac{1}{3},\frac{2}{3}\right)\right)\right] &= 0\frac{1}{3}+1\frac{2}{3} \end{aligned}\right\} = \frac{2}{3}$$

$$E\left[U_1\right] = \frac{2}{3}\frac{2}{3}+\frac{1}{3}\frac{2}{3} = \frac{2}{3}$$

# The Battle of the Sexes

- We just found out that there is no strictly profitable pure-strategy deviation

➔ There is no strictly profitable mixed-strategy deviation

- The mixed strategy NE is:

Player 1 Player 2

$$\left[\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right)\right]$$

p 1-p q 1-q

# The Battle of the Sexes

- What are the payoffs to players when they play such a mixed strategy NE?

$$u_1, u_2 = \left(\frac{2}{3}, \frac{2}{3}\right)$$

- Why are the payoffs so low?
- What is the probability for the two players not to meet?

➔ Prob(meet) = 2/3*1/3+1/3*2/3=4/9

➔ 1- Prob(meet) = 5/9 !!!

# The Battle of the Sexes

- This results seems to confirm our intuition that "magically" achieving the pure-strategy NE would be not always possible

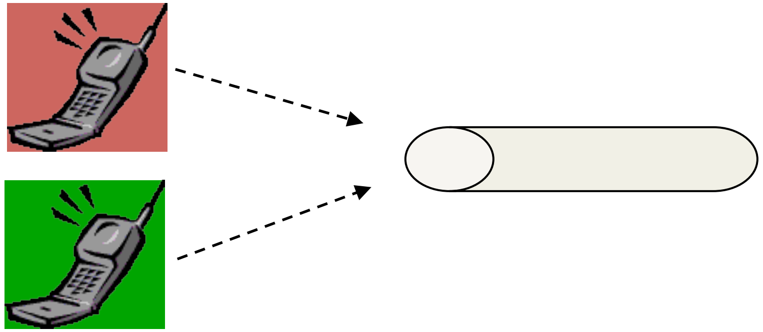
- So the real question is: why are those players randomizing in such a way that it is not profitable?

# Mixed Strategies: Interpretation

- Rather than thinking of players actually randomizing over their strategies, we can think of them ***holding beliefs*** of what the other players would play

- What we've done so far is to **find those beliefs** that make players **"indifferent"** over what they play since they're going to obtain the same payoffs

# The Multiple Access game



Time-division channel

Reward for successful transmission: 1

Cost of transmission: c
(0 < c << 1)

| Blue \ Green | Quiet | Transmit |
|---|---|---|
| Quiet | (0, 0) | (0, 1-c) |
| Transmit | (1-c, 0) | (-c, -c) |

**There is no strictly dominating strategy**

**There are two Nash equilibria**

# Mixed Strategy Nash equilibrium

**p:** probability of transmit for Blue

**q:** probability of transmit for Green

$$u_{blue} = p(1-q)(1-c) - pqc = p(1-c-q)$$

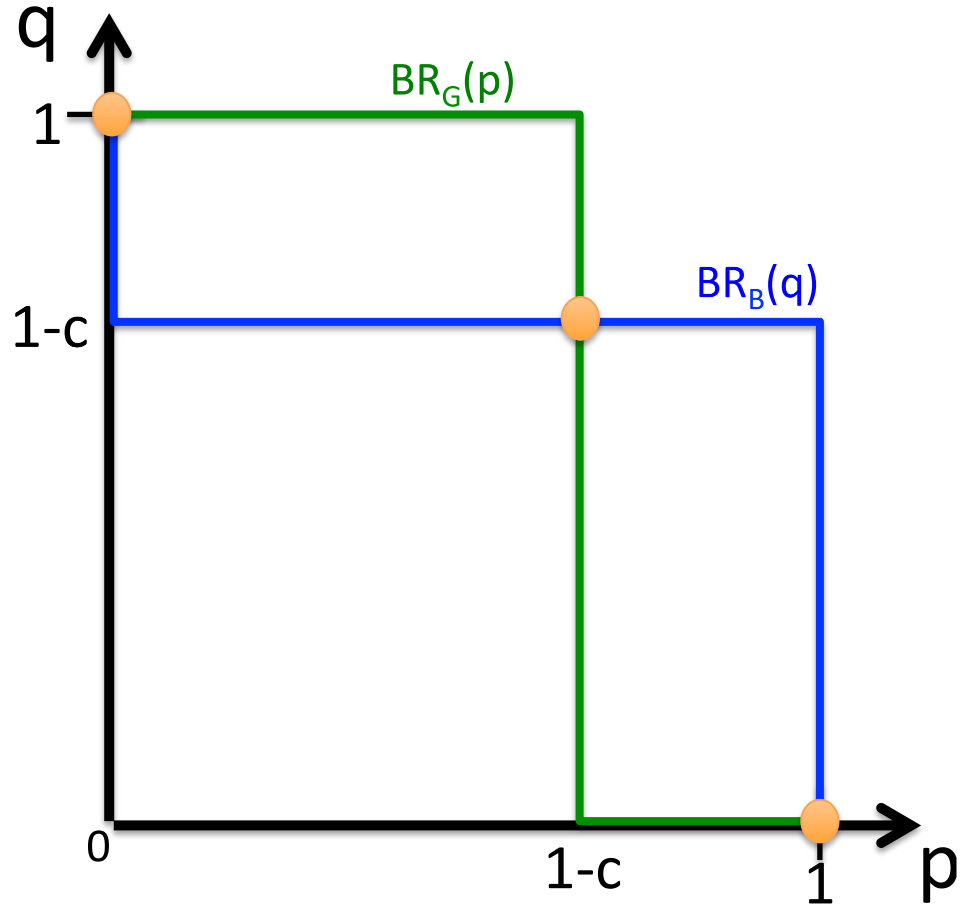
$$u_{green} = q(1-c-p)$$

objectives

- Blue: choose $p$ to maximize $u_{blue}$
- Green: choose $q$ to maximize $u_{green}$

$$p^* = 1-c,\ q^* = 1-c$$

is a Nash equilibrium

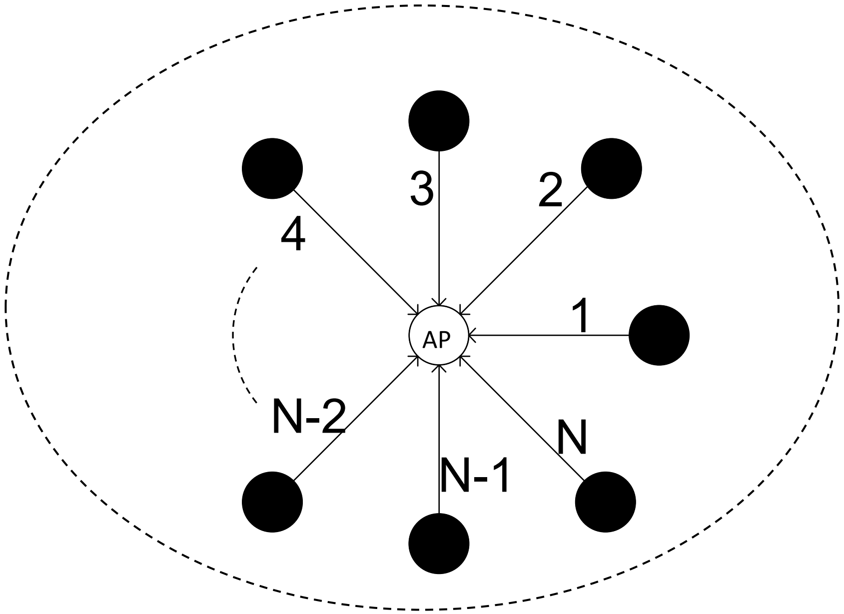


802.11 MAC Layer

# A Practical Randomization Strategy in Wireless Networks

# WiFi Networks

- N links with the same physical condition (single-collision domain):




MAC Layer → $\pi$ = Probability of Transmission

↕

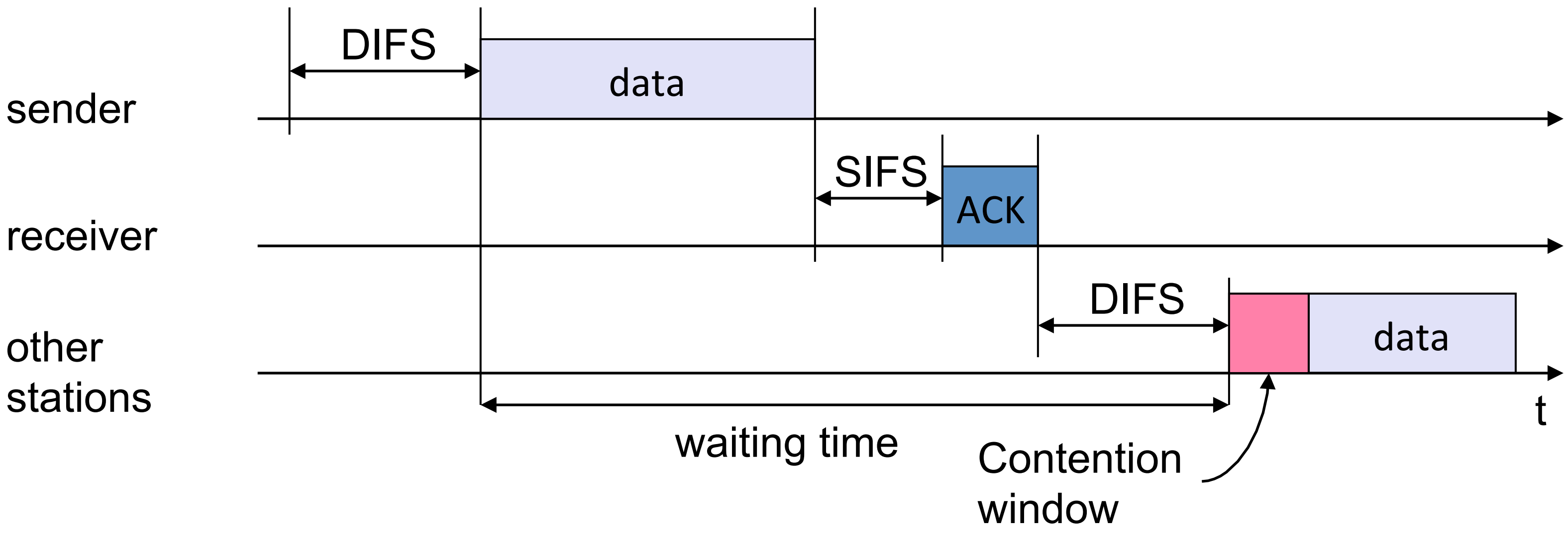
PHY Layer → P = Probability of Collision

= More than one transmission at the same time

$= 1 - (1 - \pi)^{N-1}$

# 802.11 - CSMA/CA
# A MAC Layer for WiFi Networks

- Sending unicast packets
  - station has to wait for DIFS before sending data
  - receiver acknowledges at once (after waiting for SIFS) if the packet was received correctly (CRC)
  - automatic retransmission of data packets in case of transmission errors




The ACK is sent right at the end of SIFS (no contention)

# Inter Frame Space and CW Times: Some PHY and MAC Layer Parameters

| *Parameters* | *802.11a* | *802.11b (FH)* | *802.11b (DS)* | *802.11b (IR)* | *802.11b (High Rate)* |
|---|---|---|---|---|---|
| *Slot Time* ($\mu s$) | 9 | 50 | 20 | 8 | 20 |
| *SIFS* ($\mu s$) | 16 | 28 | 10 | 10 | 10 |
| *DIFS* ($\mu s$) | 34 | 128 | 50 | 26 | 50 |
| *EIFS* ($\mu s$) | 92.6 | 396 | 364 | 205 or 193 | 268 or 364 |
| $CW_{min}(SlotTime)$ | 15 | 15 | 31 | 63 | 31 |
| $CW_{max}(SlotTime)$ | 1023 | 1023 | 1023 | 1023 | 1023 |
| *Physical Data Rate* (*Mbps*) | 6 to 54 | 1 and 2 | 1 and 2 | 1 and 2 | 1, 2, 5.5, and 11 |

# Bianchi's Model: Solution for *p* and *π*

Basically it is a system of two nonlinear equations with two variables p and π:

$$\begin{cases} p = 1 - (1-\pi)^{N-1} \\ \pi = \dfrac{2}{1 + W_{min} + pW_{min}\sum_{k=0}^{m-1}(2p)^k} \end{cases}$$

A mixing over transmission strategy

In fact π is the mixing probability of pure strategy of transmission for each mobile user

- Try to find all NE of a game between N mobile user?