بسم الله الرحمن الرحيم

Foundations of Game Theory for Electrical and Computer Engineering

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Contents

- Best Response Definition
- Best Response in Football
- Partnership Game
- Multiple Access Game

Best Response

		2	
		L	R
	U	5,1	0,2
1	M	1,3	4,1
	D	4,2	2,3

- What are the dominated strategies?

- Imagine you're player 1:
  - What would you do?

Best Response

- Would you chose U?
- What if you knew in advance that player 2 was going to chose L ?
- U would be the **best response** to L

- E.g.: your boss asks why do you choose U

➔ Given your beliefs, that was the best thing to do!!

Best Response

- Similarly, if you knew player 2 would chose R, your best response would be to play M, right?

- What if you **are not sure** what your opponent is going to play?

Best Response

		2	
		L	R
	U	5,1	0,2
1	M	1,3	4,1
	D	4,2	2,3

- Suppose that you believe that 2 plays L with probability 0.5
  - Expected payoff for playing U vs. 50% L , 50% R
    0.5*5 + 0.5* 0 = 2.5
  - Expected payoff for playing M vs. 50% L , 50% R
    0.5*1 + 0.5* 4 = 2.5
  - Expected payoff for playing D vs. 50% L , 50% R
    0.5*4 + 0.5* 2 = 3

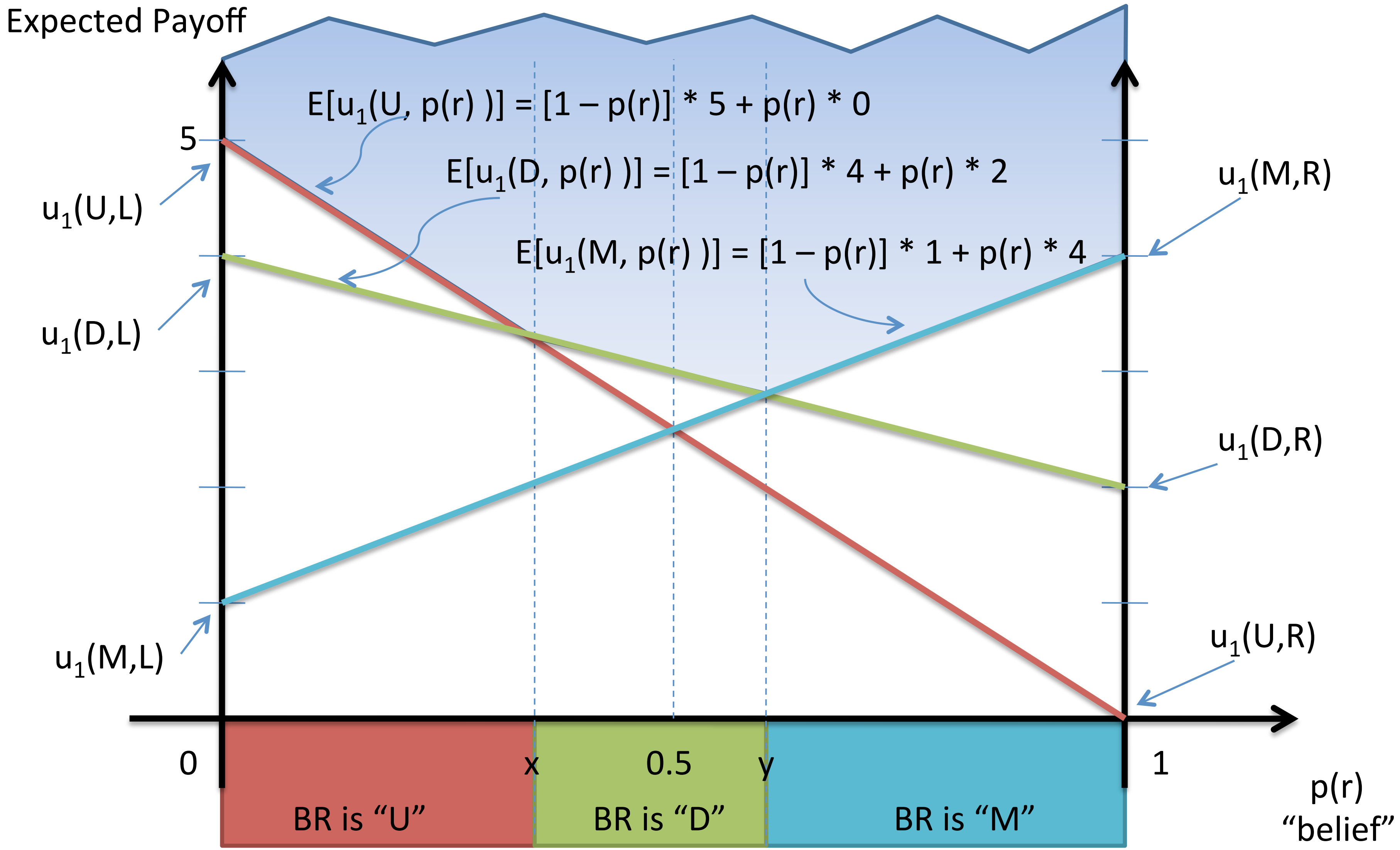
It turns out that D is the **best response**,
when there's an equal probability that your opponent will play l or r.

Best Response

- Obviously, the 50% L - 50% R is just a belief
- I could believe my opponent would lean to left, e.g. with a 75% L - 25% R probabilities

- Can we use a representation to sum up all these possibilities and come up with a prediction?

Best Response Functions



Summary

- We introduced the idea of **best response** (BR)

➔do the best you can do, given your belief about what the other players will do

- We saw a simple game in which we applied the BR idea and worked with plots

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Penalty Kick Game

Goalkeeper

		l	r
	L	4, -4	9, -9
kicker	M	6, -6	6, -6
	R	9, -9	4, -4

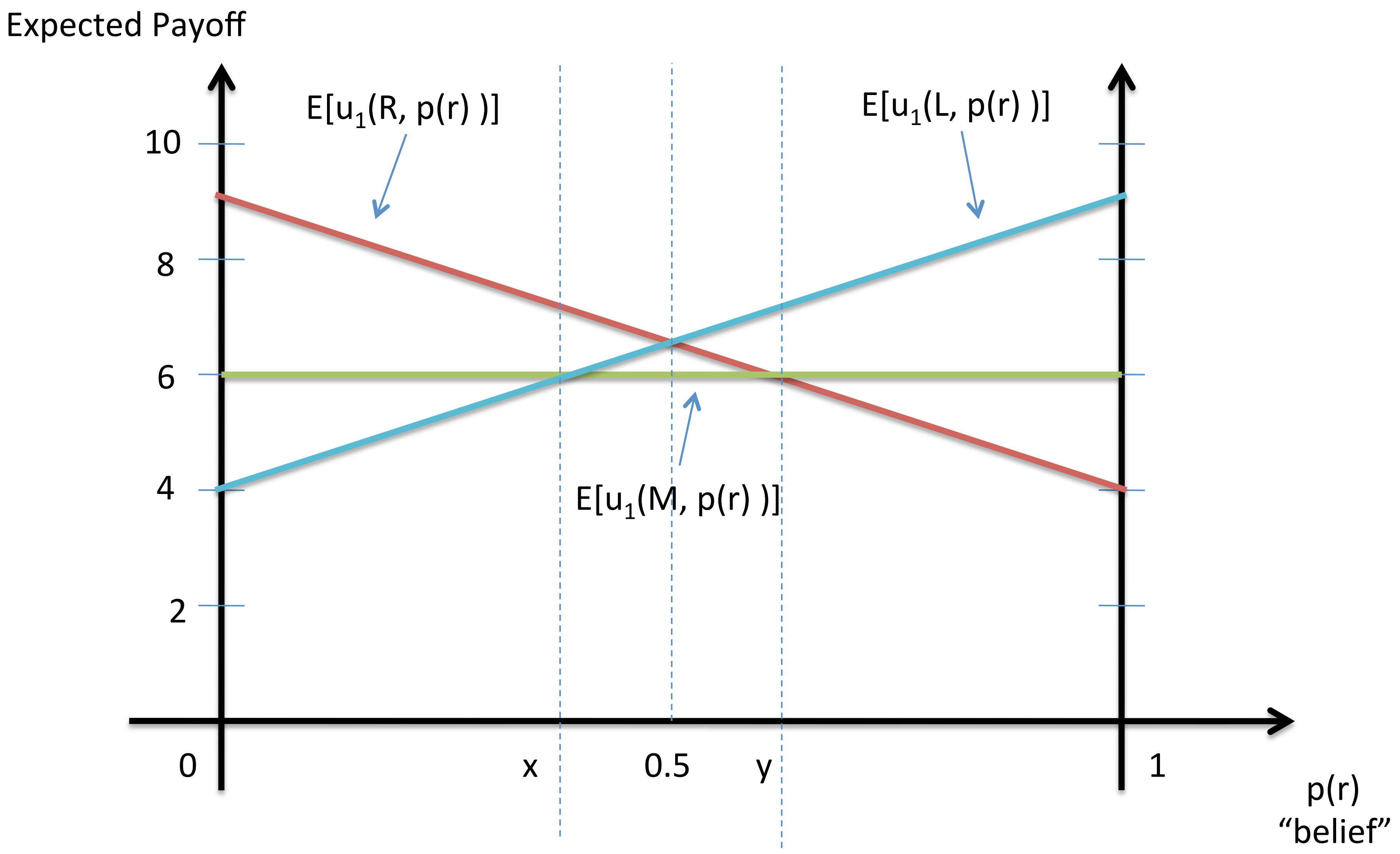
- Payoffs approximate the probabilities of scoring for the kicker, and the negative of that for the goalie
- Assumption: we ignore the "stay put" option for the goalie
- Example:
  - $u_1(L,l) = 4$ → 40% chance of scoring
  - $u_1(L,r) = 9$ → 90% chance of scoring

Penalty Kick Game

- What would you do here?
- Is there any dominated strategy?

- If we stopped to the idea of iterative deletion of dominated strategies, we would be stuck!

- If you were the kicker, were would you shoot?


Expected Payoff
E[u$_1$(R, p(r))]
E[u$_1$(L, p(r))]
E[u$_1$(M, p(r))]
10
8
6
4
2
0
x
0.5
y
1
p(r)
"belief"

Penalty Kick Game

- What's the lesson here?
- Assume for a moment these numbers are true

- If the goalkeeper is jumping to the right with a probability less than 0.5, then you should shoot ….

Lesson: Don't shoot to the middle

Main Lesson

Do not choose a strategy that is never a BR to any belief*

* any means all probabilities

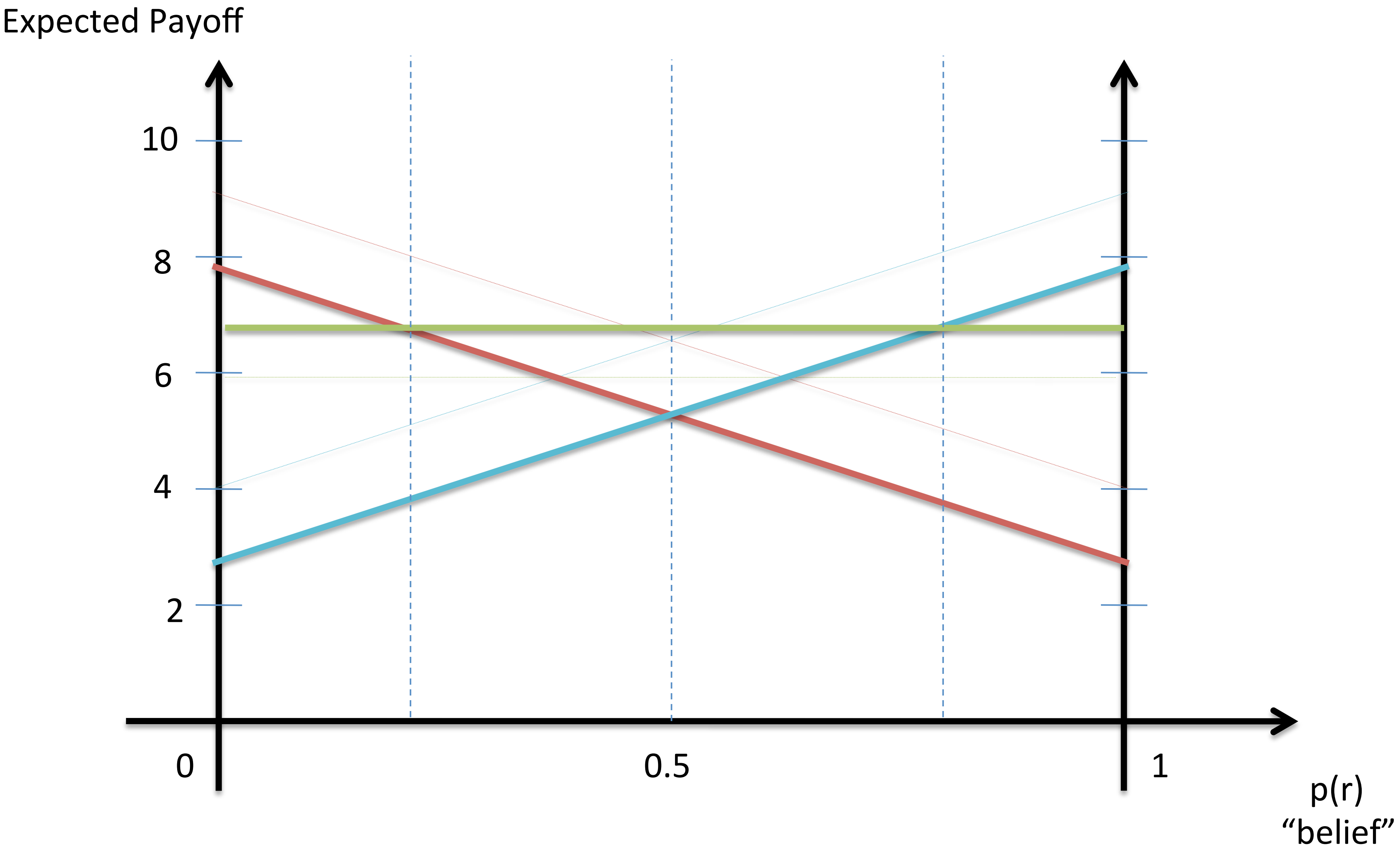
Penalty Kick Game

- Notice how we could eliminate one strategy even though nothing was dominated
  - With deletion of dominated strategies we got nowhere
  - With BR, we made some progress…

- Can we do better? What are we missing here?

Penalty Kick Game: Some Extensions

- **Right footed players** find it easier to shoot to their left!
- The goalie might **stay in the middle**
- The probabilities we used before are artificial, **what about reality**?
- What about considering also the **speed**?
- And the **precision**?



See what happens? If you are less precise but strong you’d be better off by shooting to the middle

Definition

Definition: Best Response

Player i's strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$u_i(\hat{s}_i\ ,\ s_{-i}) \geq u_i(s'_i\ ,\ s_{-i}) \text{ for all } s'_i \text{ in } S_i$$

or

$$\hat{s}_i \text{ solves } max\ u_i(s_i\ ,\ s_{-i})$$

Reminder!

Definition: Strict dominance

We say player i's strategy s_i' is strictly dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for all } s_{-i}$$

Definition

Definition: Best Response (general)

Player i’s strategy $\hat{s}_i$ is a BR to the belief p about the others’ choices if:

$$E[\ u_i(\hat{s}_i\ ,\ p)\ ] \geq E[\ u_i(s'_i\ ,\ p)\ ] \text{ for all } s'_i \text{ in } S_i$$

or

$$\hat{s}_i \text{ solves } max\ \{\ E\ [\ u_i(s_i\ ,\ s_{-i}),p\ ]\ \}$$

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The Partnership Game

- Two individuals (players) who are going to supply an input to a joint project
- The two individuals **share 50%** of the profit
- The two individuals supply efforts individually

- Each player chooses the **effort level** to put into the project (e.g. working hours)

The Partnership Game

- Let’s be more formal, and normalize the effort in hours a player chooses

- Si = [0,4]

➔Note: this is a continuous set of strategies

The Partnership Game

- Let's now define the profit to the partnership

$$\text{Profit} = 4\,[s_1 + s_2 + b\, s_1\, s_2]$$

- Where:
  - s_i = the effort level chosen by player i
  - b = Synergy / Complementarity
  - $0 \leq b \leq 1/4$
- **Why** is there the term $s_1\, s_2$?

The Partnership Game

- What's missing? Payoffs!

$$u_1(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\ s_1\ s_2)] - s_1^2$$

$$u_2(s_1 , s_2) = ½ [4 (s_1 + s_2 + b\ s_1\ s_2)] - s_2^2$$

- That is:
  - Players share the profit in half
  - They bear a cost proportional to the square of their effort level
  - Note: **payoff = benefit - cost**

The Partnership Game

- Let's analyze this game with the idea of BR

- But how can we draw a graph with a continuous set of strategies?

- Recall the definition of best response

The Partnership Game

Definition: Best Response

Player i's strategy $\hat{s}_i$ is a BR to strategy s_{-i} of other players if:

$$\hat{s}_i = \arg\max u_i(s_i , s_{-i})$$

- We are going to use some calculus here

$$\hat{s}_1 = \textbf{arg max} \{ 2 (s_1 + s_2 + b\, s_1\, s_2) - s_1{}^2 \}$$

The Partnership Game

- So we differentiate:
- ***F.o.d.*** : $2\ (1 + b\ s_2) - 2s_1 = 0$
- ***S.o.d.*** : $-2 < 0$

$$\hat{s}_1 = 1 + b\ s_2 \quad = BR_1(s_2)$$
$$\hat{s}_2 = 1 + b\ s_1 \quad = BR_2(s_1)$$

➔ due to symmetry of the game

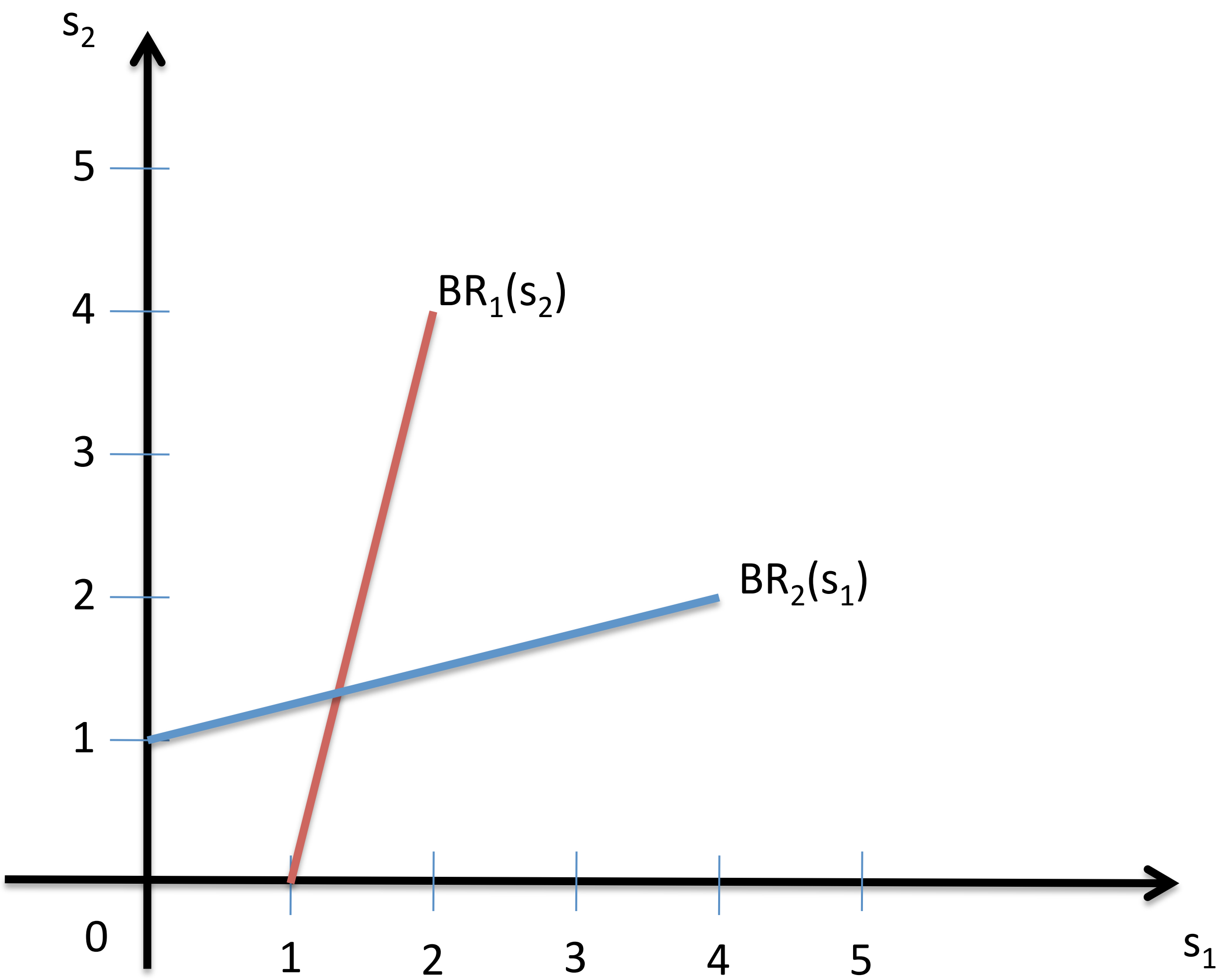
The Partnership Game

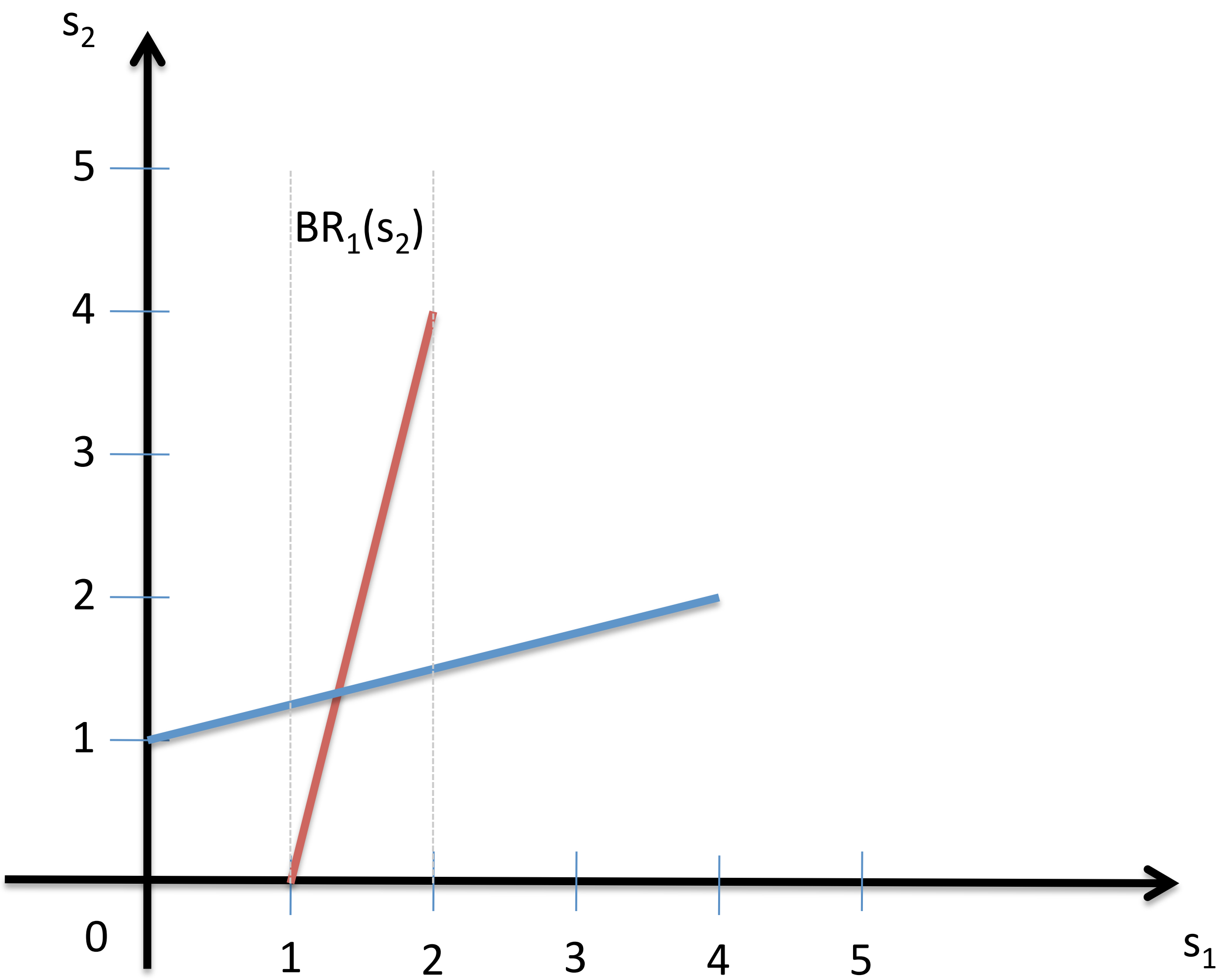
- Alright, we have the expressions that tell me:

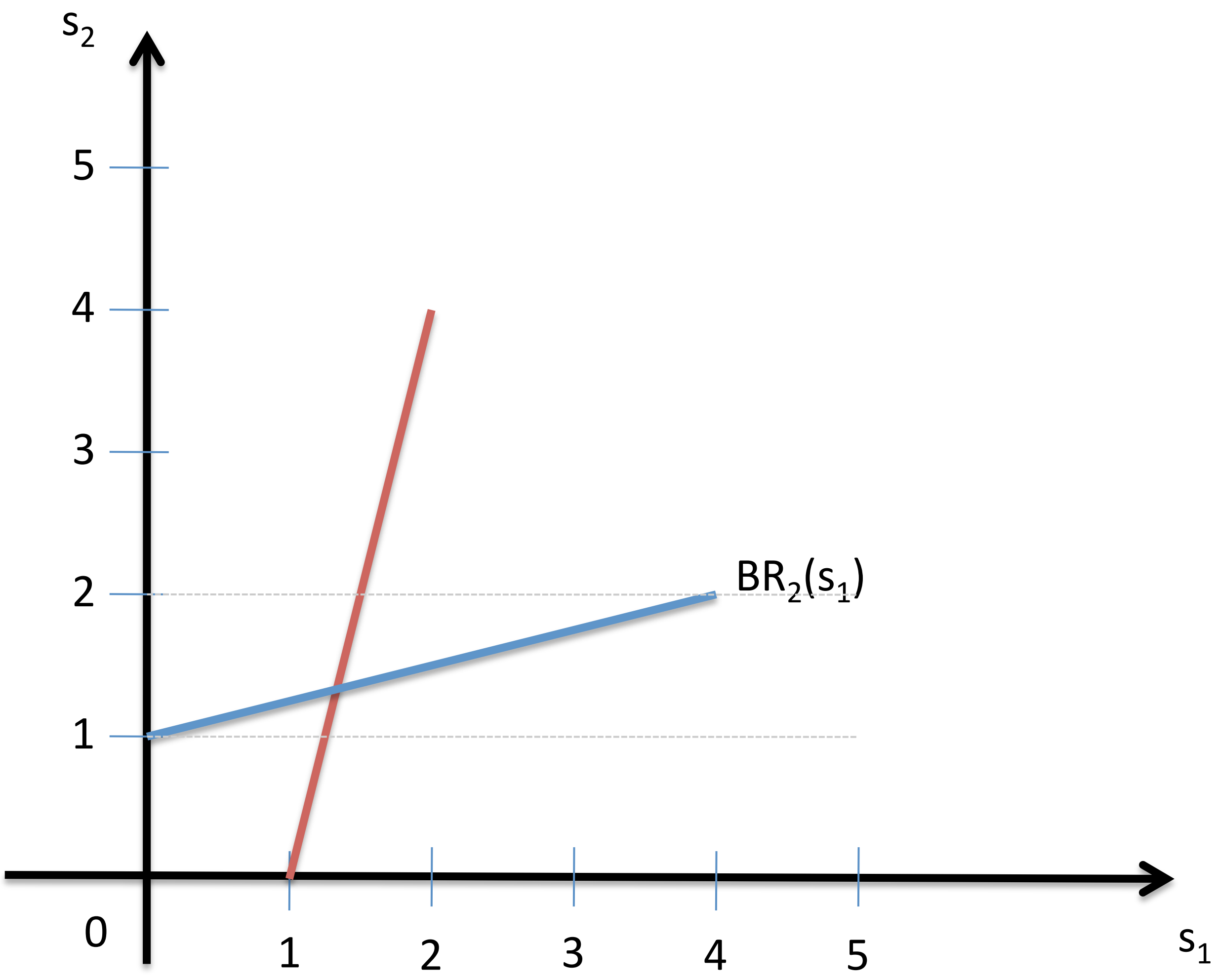
➔player *i* best response, given what player *j* is doing

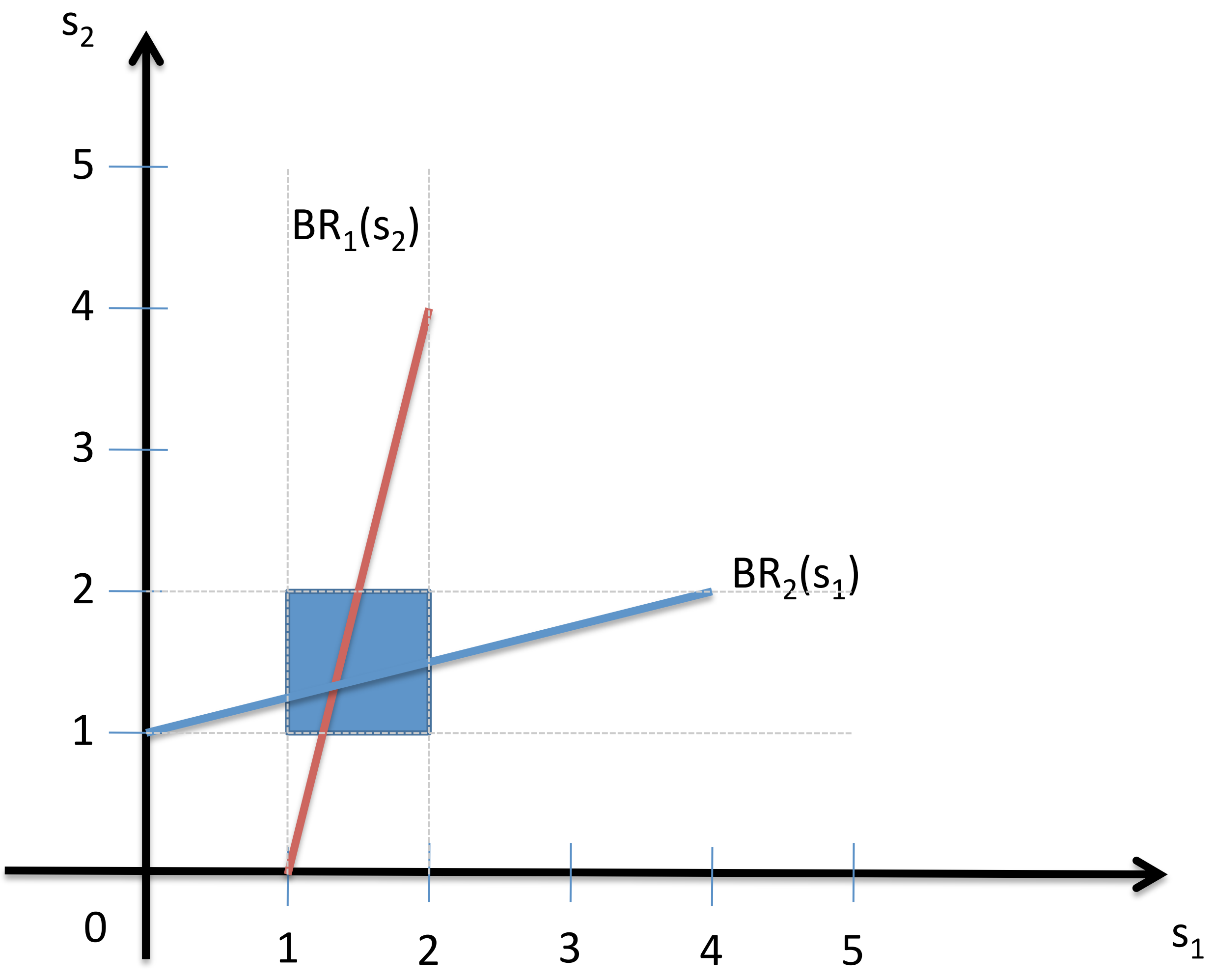
- Now, let's draw the two functions we found and have a look at what we can say
- Let's also fix the only parameter of the game:

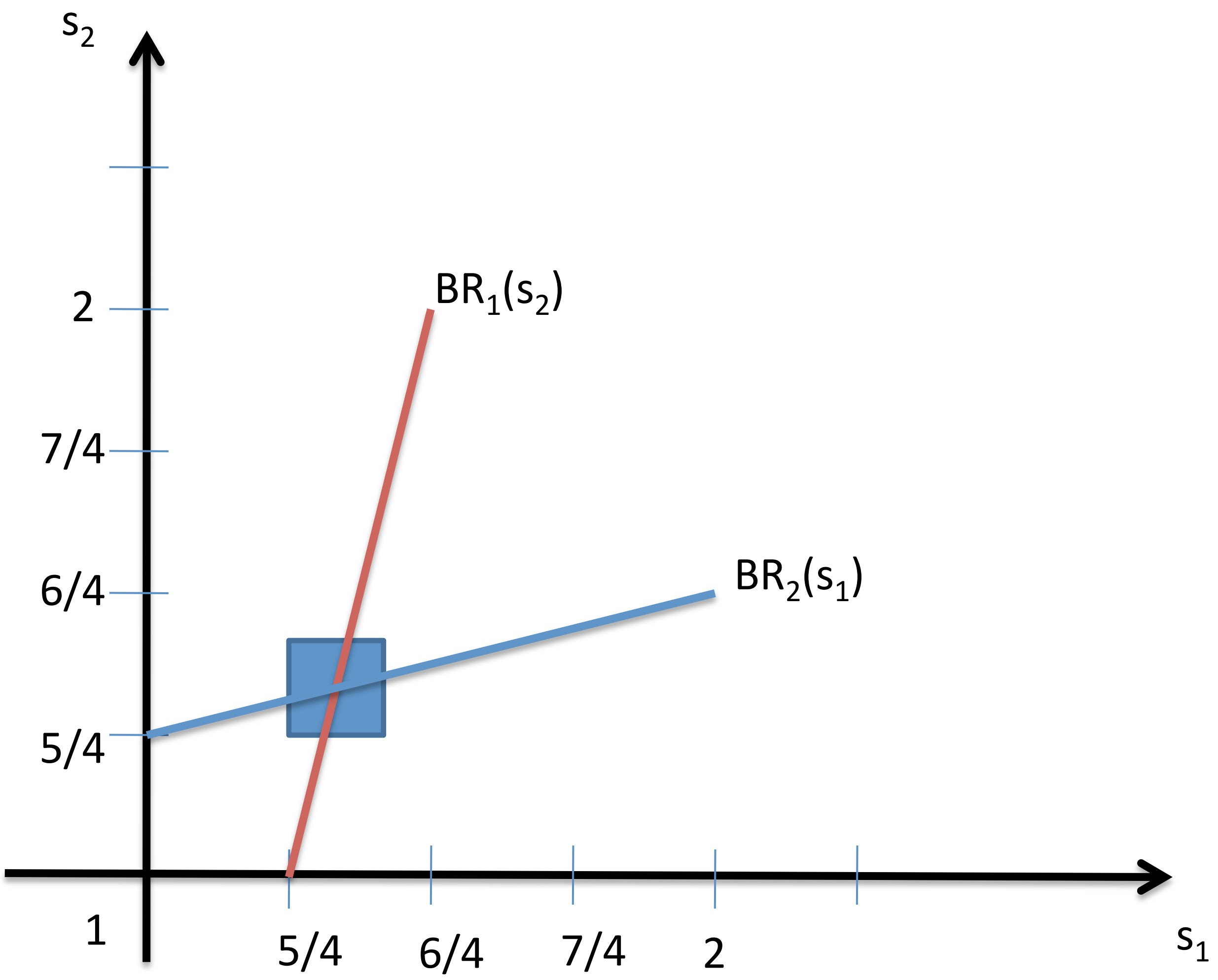
$$b = 1/4$$


s_2
5
4
3
2
1
0
$BR_1(s_2)$
$BR_2(s_1)$
1
2
3
4
5
s_1


s_2
5
4
3
2
1
0
$BR_1(s_2)$
1
2
3
4
5
s_1


s_2
5
4
3
2
1
0
1
2
3
4
5
s_1
$BR_2(s_1)$


s_2
5
4
3
2
1
0
$BR_1(s_2)$
$BR_2(s_1)$
1
2
3
4
5
s_1


s_2
2
7/4
6/4
5/4
1
$BR_1(s_2)$
$BR_2(s_1)$
5/4
6/4
7/4
2
s_1

The Partnership Game

- We started with a game
- We found what player 1 BR was for every possible choice of player 2
- We did the same for player 2
- We eliminated all strategies that were never a BR
- We looked at the ones that were left, and eliminated those that were never a best response
- …
- **Where are we going to?**

The Partnership Game

$$s^*_1 = 1 + b\ s_2$$

$$s^*_2 = 1 + b\ s_1$$

The intersection ➔ $s^*_1 = s^*_2$

➔ $s^*_1 = 1/(1\text{-}b)$

The Partnership Game

We came up with a prediction
on the effort levels

Question: is the amount of work we found previously a good amount of work?

Question: are the players working more or less than an efficient level?

The Partnership Game

- Why is it that in a joint project we tend to get inefficiently little effort when we figure out what's the best response in the game?

- NOTE: this is not a dilemma situation
  - Why?

The Partnership Game

- The problem is not really the amount of work
- Also, the problem is not about synergy, i.e., the factor b

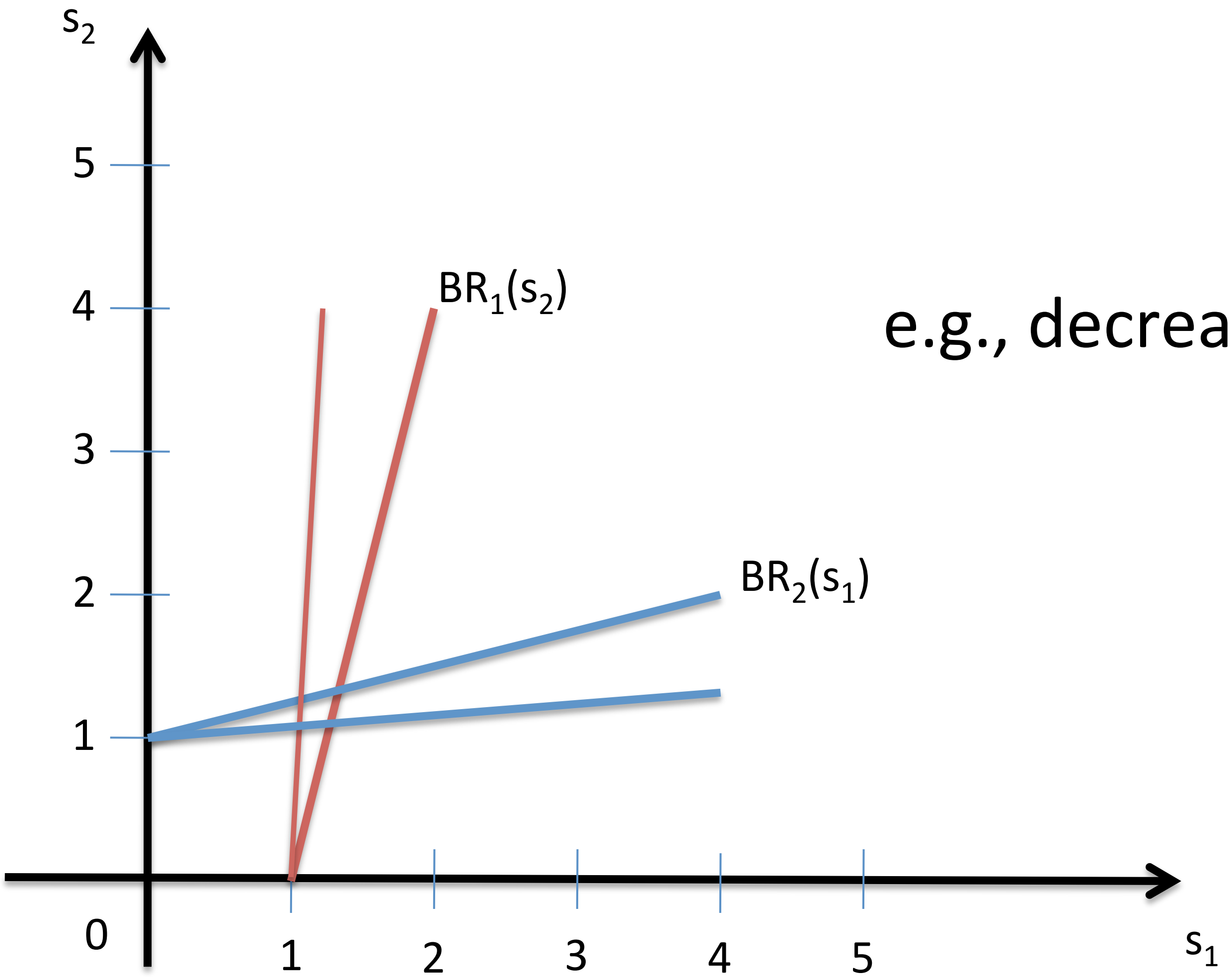
- The problem is that at the margin, I bear the cost for the extra unit of effort I contribute, but I'm only reaping half of the induced profits, because of profit sharing

- This is known as an "externaility"

➔In other words, my effort benefits my partner, not just me

The Partnership Game

By the way, how would the situation change by varying the only parameter of the game?



e.g., decrease b

Informally, what we have done so far is to determine the **Nash Equilibrium** of the game

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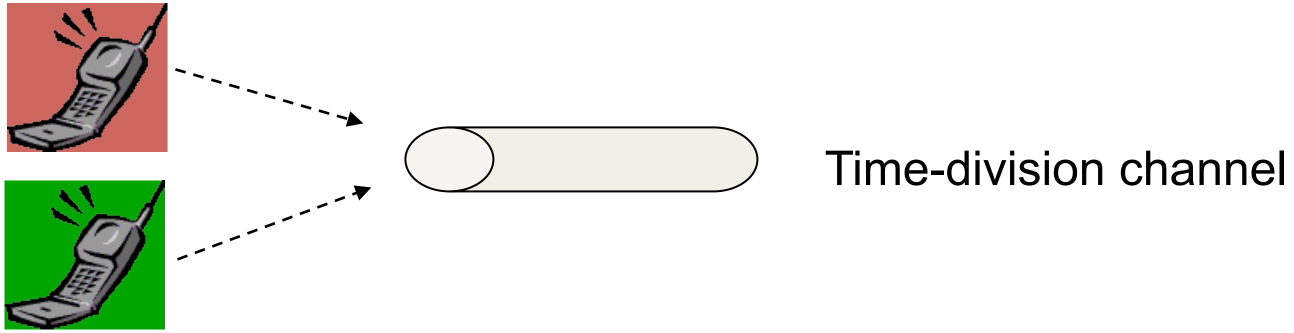
The Multiple Access game



Reward for successful transmission: 1

Cost of transmission: c
(0 < c << 1)

Blue \ Green	Quiet	Transmit
Quiet	(0, 0)	(0, 1-c)
Transmit	(1-c, 0)	(-c, -c)

There is no strictly dominating strategy

What is the best response?

Best Response Functions

q: probability of transmit for Green

$$E\{u_{blue},T\}=(1-q)(1-c)-qc=(1-c)-q$$

$$E\{u_{blue},Q\}=0$$