بسم الله الرحمن الرحیم

Foundations of Game Theory for Electrical and Computer Engineering

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Pick a Number Game

Without showing your neighbor what you're doing, write down an integer number between 1 and 100. I will calculate the average number chosen in the class. The winner in this game is the person whose number is closest to two-thirds (2/3) of the average in the class. The winner will win 10 $ minus the difference in cents between her choice and that two-thirds of the average.

Example: 3 students

Numbers: 25, 5, 60

Total: 90, Average: 30, 2/3*average: 20

25 wins: 10 $ –.01 * 5 = 9.95 $

Notations

	Notation	Pick a Number Game
Players	i, j, …	You all
Strategy	s_i: a particular strategy of player i s_{-i}: the strategy of everybody else except player i	S_4=12, s_8=22
Strategy Set	S_i: the set of possible strategies of player i	{1, 2, …, 100}
Strategy Profile	s: a particular play of the game "strategy profile" (vector, or list)	The collection of your pieces of paper
Payoffs	$u_i(s_1,\ldots, s_i,\ldots, s_N) = u_i(s)$	$u_i(s) = \begin{cases} \$10 - .01*\Delta & \textit{if you win} \\ 0 & \textit{otherwise} \end{cases}$

Assumptions

- We assume all the ingredients of the game to be known
  - Everybody knows the possible strategies everyone else could choose
  - Everybody knows everyone else's payoffs

Complete Information Game

- This is not very realistic, but we start from this class of games

Classification of games

Non-cooperative	Cooperative
Static	Dynamic (repeated)
Strategic-form	Extensive-form
Perfect information	Imperfect information
Complete information	Incomplete information

Perfect information: each player can observe the action of each other player.

Complete information: each player knows the identity of other players and, for each of them, the payoff resulting of each strategy.

Example

		Player 2	
	L	**C**	**R**
Player 1 **T**	5, -1	11, 3	0,0
B	6, 4	0, 2	2,0

Players	**1, 2**	
Strategy sets	S_1={T,B}	S_2={L,C,R}
Payoffs	U_1(T,C) = 11	U_2(T,C) = 3

NOTE: This game is not symmetric

Game Analysis

- How is the game going to be played?
- Does player 1 have a dominated strategy?
- Does player 2 have a dominated strategy?

- For a strategy to be dominated, we need another strategy for the same player that does always better (in terms of payoffs)

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Definition: Strict dominance

We say player i's strategy s_i' is strictly dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for all } s_{-i}$$

No matter what other people do, by choosing s_i instead of s_i', player i will always obtain a higher payoff.

“Hannibal” game

- An invader is thinking about invading a country, and there are **2 ways through** which he can lead his army.
- You are the defender of this country and you have to decide **which of these ways you choose to defend**: you can only defend one of these routes.
- **One route is a hard pass:** if the invader chooses this route he will **lose one battalion** of his army (over the mountains).
- If the invader **meets your army**, whatever route he chooses, he will **lose a battalion**

“Hannibal” game

		Attacker	
		e	h
Defender	E	1, 1	1, 1
	H	0, 2	2, 0

Strategies

1. e, E = Easy Path ;
2. h,H = Hard Path

Payoffs:

1. **Attacker:** Number of battalions in your country
2. **Defender:** Number of attacker’s lost battalions

“Hannibal” game

- You’re the defender: what would you do?
- Is it true that defending the easy route dominates defending the hard one?

- You’re the attacker: what would you do?

- Now, what the defender should do, if he would put himself in the attacker shoes?

Definition: Weak dominance

We say player i's strategy s_i' is weakly dominated by player i's strategy s_i if:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_{-i}$$
$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for some } s_{-i}$$

No matter what other people do, by choosing s_i instead of s_i' , player i will always do **at least as well**, and in some cases she does strictly better.

It turns out that, historically, Hannibal chose H!

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Example: 3 students

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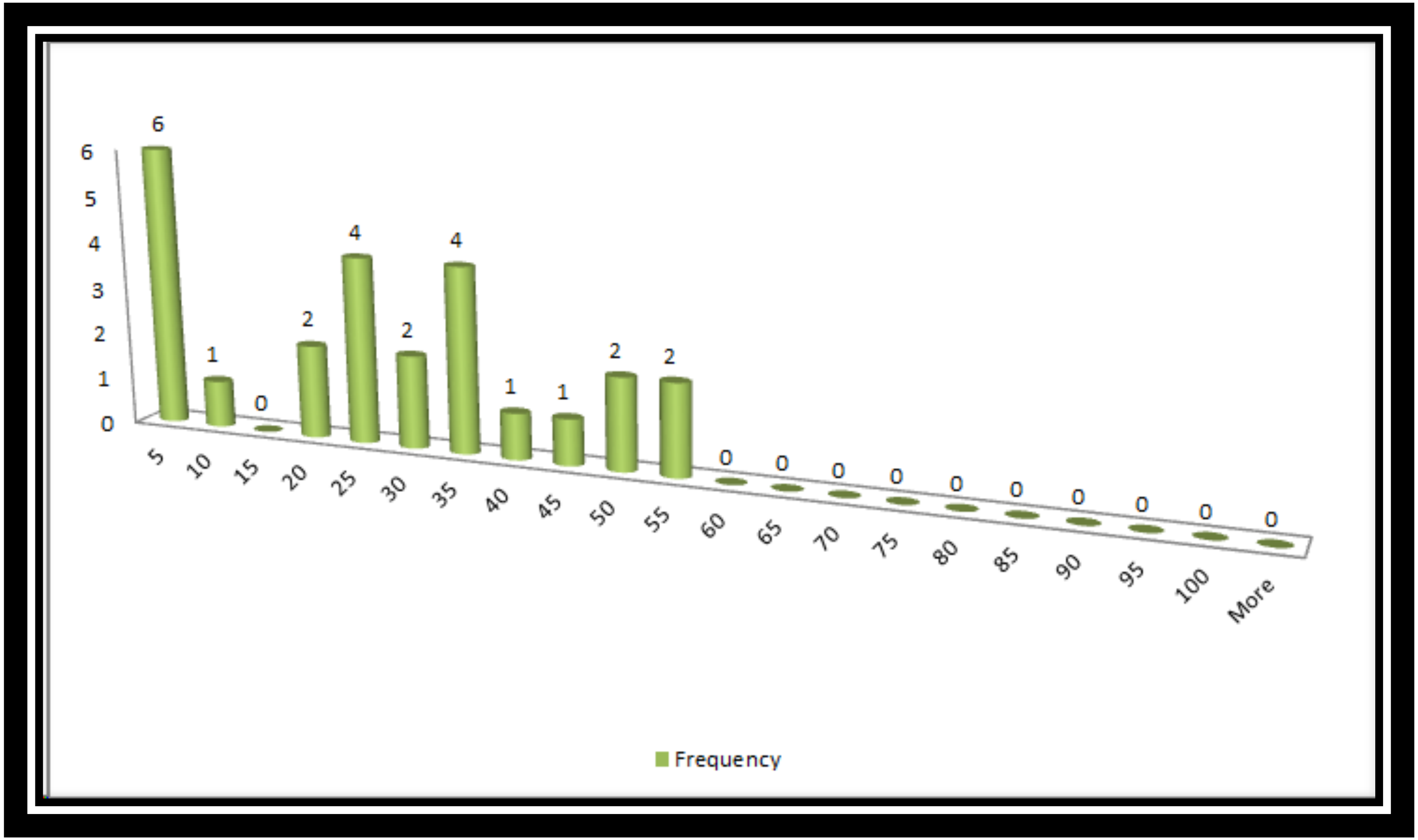
What did you do?

- What we know:
  - Do not choose a strictly dominated strategy
  - Also, do not choose a weakly dominated strategy
  - You should put yourself in others' shoes, try to figure out what they are going to play, and respond appropriately

How did you play?



First Clue!

- A possible assumption:
  - – People chose numbers uniformly at random
  - ➔The average is 50
  - ➔2/3 * average = 33.3

- What's wrong with this reasoning?

IUT students are not rand()!

Dominated Strategy?

- Let's try to find out whether there are dominated strategies

- If everyone would chose 100, then the winning number would be 67

➔Numbers bigger than 67 are weakly dominated by 67

➔Rationality tells not to choose numbers bigger than 67

New Game!

- So now we've eliminated dominated strategies, it's like a new game played over the set [1, …, 67]
- Once you figured out that nobody is going to choose a number above 67, the conclusion is

Also strategies above 45 are ruled out

- This means:
  1. Rationality
  2. Knowledge that others are rational as well

- Note:

They are weakly dominated, only once we delete 68-100

Iterative Deletion

- Eventually, we can show that also strategies above 30 are weakly dominated, once we delete previously dominated strategies

- We can go on with this line of reasoning and end up with the conclusion that:

- 1 was the winning strategy!

Common Knowledge

- **Common knowledge**: you know that others know that others know … and so on that rationality is underlying all players' choices

Theory vs. Practice

- **Q:** Why was it that 1 wasn't the winning answer?

- **A:** We need a strong assumption, that is that all players are rational and they know that everybody else's rational as well

Common Knowledge

Rationality

Rationality and Knowledge of Other's Rationality

Rationality, Knowledge of Other's Rationality, and Knowledge of Knowledge of Rationality (know that you know that I know ….)

Rationality, Knowledge of Other's Rationality, Knowledge of Knowledge of Rationality, and Knowledge of knowledge of knowledge of Rationality

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Our Game Results: 2015

– Average number was: 24.76 (**2014:** 35.7, **2013:** 35.78, **2012:** 26.14)

– Winning number was: 2/3XAverage = 16.51 (**2014:** 23.80, **2013:** 23.85, **2012:**17.43)

Summary

- We've explored a bit the idea of deleting dominated strategies
  - Look at a game
  - Figure out which strategies are dominated
  - Delete them
  - Look at the game again
  - Look at which strategies are dominated now
  - … and so on …

Summary

- **Iterative deletion of dominated strategies** seems a powerful idea, but it's also dangerous if you take it literally
- In some games, iterative deletion converges to a single choice, in others it may not (we'll see shortly an example)
- HINT: try to identify all dominated strategies at once before you delete, this may save you some rounds…

Let us play
the same Game, again!

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1. Formal Definitions/Notations
2. Strict versus Weak Dominance
3. Iterative Deletion of Dominated Strategy
4. Median Voter Theorem
5. Model Simplification for Engineering Application: Examples

Election Game Model

- **2 candidates**
- Choosing their political positions on a **spectrum**
- Assume the spectrum has 10 positions

1 2 3 4 5 6 7 8 9 10

LEFT WING RIGHT WING

Election Game Model

- There are 10% of the voters at each of these positions:
  - Voters are uniformly distributed
- Voters will eventually vote for the closest candidate (i.e., for the candidate whose position is closest to their own)
- We break ties by splitting votes equally

Votes:	10%	10%	10%	10%	10%	10%	10%	10%	10%	10%
	1	2	3	4	5	6	7	8	9	10

LEFT WING RIGHT WING

Election Game Strategies

- We assume payoffs follow the idea that the candidates aim to **maximize** their share of vote (Win the Election)

- Are there any dominated strategies here?

Election Game
Analysis of Dominated Strategy

- Is position 1 dominated? If so, what dominates it? Let's test, e.g. how is 1 vs. 2

s_{-i}	**1's Payoff for s_i=1**		**1's Payoff for s_i=2**
Vs. 1	$u_1(1,1) = 50\ \%$	<	$u_1(2,1) = 90\%$
Vs. 2	$u_1(1,2) = 10\ \%$	<	$u_1(2,2) = 50\%$
Vs. 3	$u_1(1,3) = 15\ \%$	<	$u_1(2,3) = 20\%$
Vs. 4	$u_1(1,4) = 20\ \%$	<	$u_1(2,4) = 25\%$
…	…	…	….

- Do you see a pattern coming up here?

➔We conclude that 2 **strictly dominates** 1

- We're not saying that 2 wins over 1

Election Game
Analysis of Dominated Strategy

- Using a similar argument, we have that:

➔9 **strictly dominates** 10

- Is there anything else dominated here?
- What about 2 being dominated by 3?

Vs. 1	**$U_1(2,1)$ = 90 %**	**>**	**$U_1(3,1)$ = 85%**

Election Game
Analysis of Dominated Strategy

- Even though 2 is not a dominated strategy, if we do the process of iterative deletion and delete dominated strategies (1 and 9)…
- Would 3 dominate 2?

Vs. 2	**$u_1(2,2) = 50\ \%$**	**<**	**$u_1(3,2) = 80\%$**
Vs. 3	$u_1(2,3) = 20\ \%$	<	$u_1(3,3) = 50\%$
Vs. 4	$u_1(2,4) = 25\ \%$	<	$u_1(3,4) = 30\%$
Vs. 5	$U_1(2,5) = 30\ \%$	<	$u_1(3,5) = 35\%$
…	…	…	….

Election Game
Analysis of Dominated Strategy

- Strategies 2 and 8 are **not** dominated

➔They are dominated once we realize that strategies 1 and 10 won't be chosen

- If we continued the exercise, where would we get?

Election Game Result

- It turns out that 5 and 6 are not dominated
- What's the prediction that game theory suggests here?

➔Candidates will be squeezed towards the center, they're going to chose positions very close to each other

In political science this is called the
Median Voter Theorem

Election Game: Similar Examples

- The same model has applications in economics as well (and computer science): **product placement**
- **Example: Placing Gas Stations (abroad) or banks (here!)**
  - Spread themselves evenly out over the town
  - On every road
- As we all know, this doesn't happen:
  All gas stations tend to crowd into the same corners, all the fast foods crowd as well, … you name it

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Model Simplification

- We have been using a model of a real-world situation, and tried to predict the outcome using game theory

- What is missing? Is there anything wrong with the model?

Simplification in Median Voter

- Voters are not evenly distributed
- Some people do not vote
- There may be more than 2 candidates
- There may be higher "dimensions" to the problem

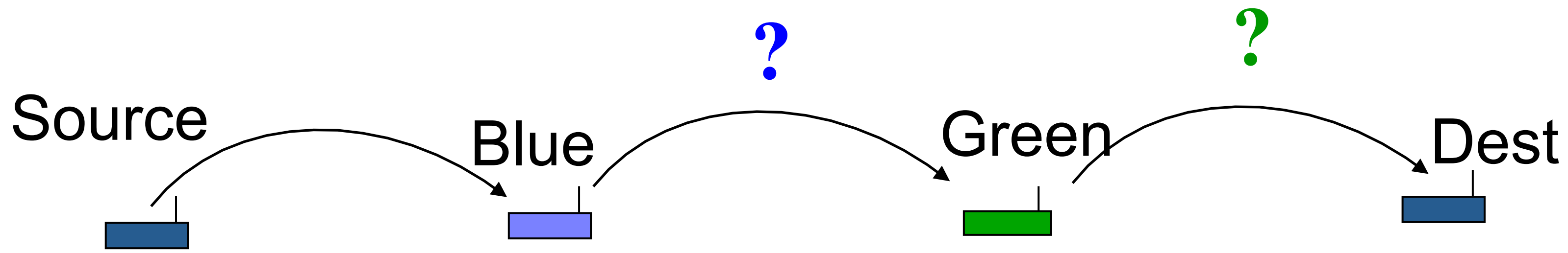
Model Simplification

- So if we're missing so many things, our model is useless, and in general modeling (as an abstraction effort) is useless!!
- No: first, analyze a problem with simplifying assumptions, then relax them and see what happens
- E.g.: would a different voters distribution change the result?

Model Simplification: Engineering Approach

- We basically make lots of abstractions in make game theoretical models for our engineering problems
- Not a bad idea to start with abstraction, but you must be careful about what you design

The Joint Packet Forwarding Game



- Reward for packet reaching the destination: 1
- Cost of packet forwarding: c (0 < c << 1)

Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 0)
Drop	(0, 0)	(0, 0)

No strictly dominated strategies !

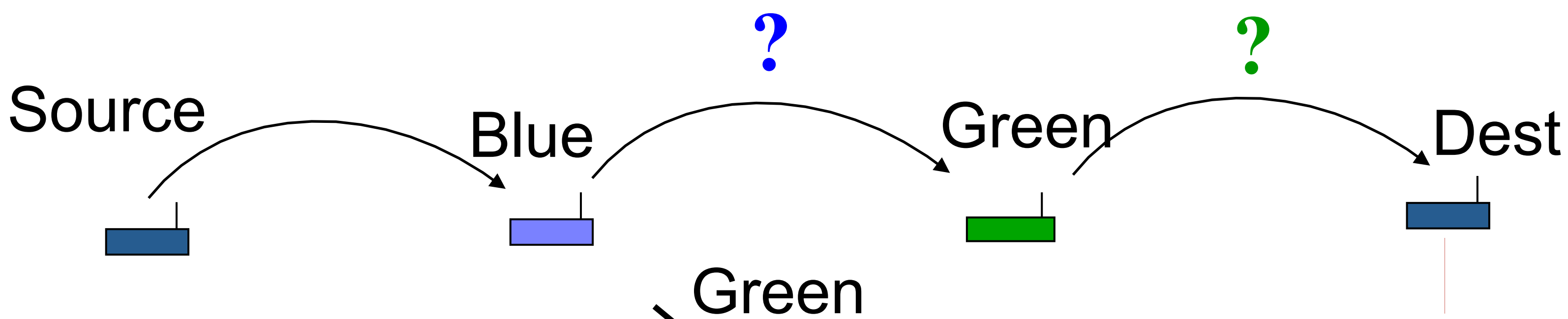
Weak dominance

Weak dominance: strictly better strategy for at least one opponent strategy

Strategy s_i is weakly dominates strategy s'_i if:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_{-i} \in S_{-i}$$

with strict inequality for at least one s_{-i}



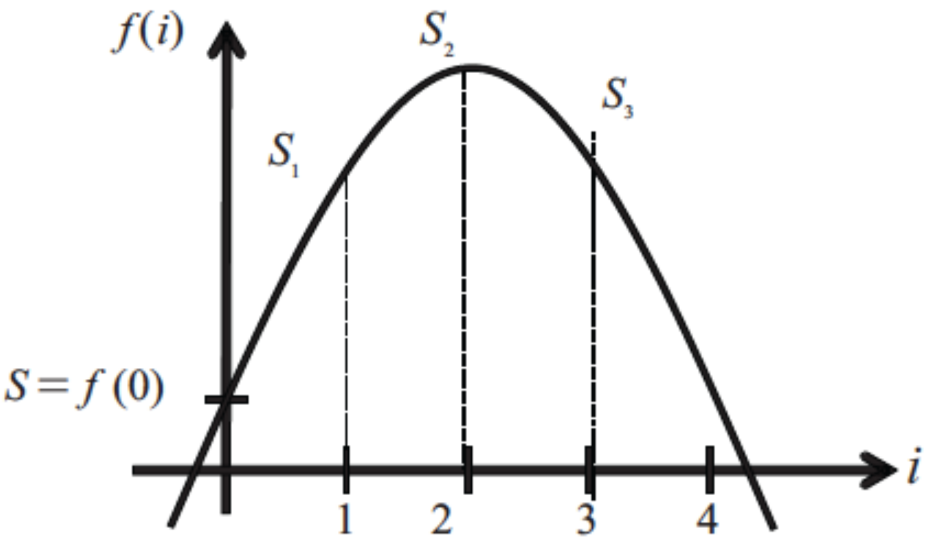
Blue \ Green	Forward	Drop
Forward	(1-c, 1-c)	(-c, 0)
Drop	(0, 0)	(0, 0)

Iterative weak dominance

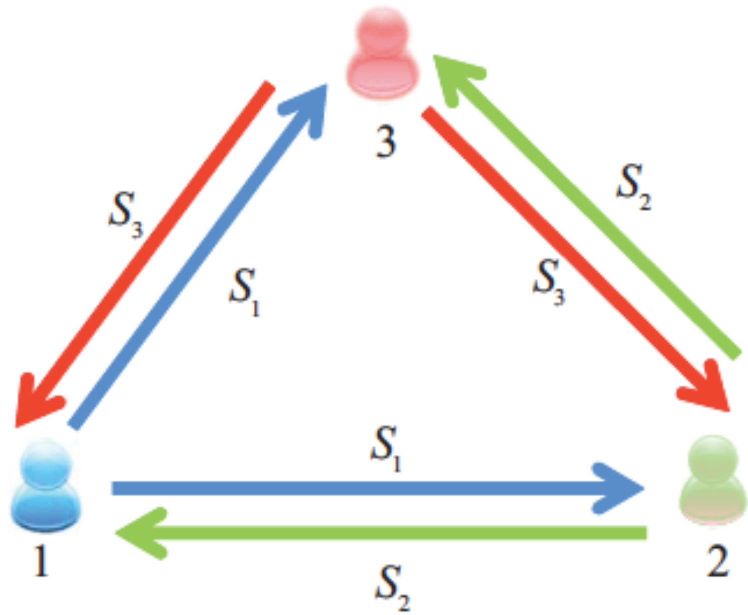
BUT

The result of the iterative weak dominance is not unique in general !

Weak Dominance in Threshold Cryptography



$S = f(0)$ is the secret, and each S_i is calculated using a polynomial function



Each party should receive the other two secret shares to calculate the secret.

J. Halpern and V. Teague, "Rational secret sharing and multiparty computation" In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing, 2004*

Weak Dominance in Threshold Cryptography

- The parties are rational and that they cooperate if it is in their interest to share a part of the secret (it increases its payoff)
- Given the rationality assumption:

  "Rational parties will not broadcast their shares"
- Not sending the share (Defect) is a weakly dominating strategy in the game between the parties
- Results make sense if we consider that the parties have common knowledge about the running time of the protocol